Emmanuel Candes Reading Group
Towards a Mathematical Theory of Super-Resolution
E. Candes C. Fernandez-Granda [2012]

Philippe Weinzaepfel

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Goal

Enhancing the resolution of a sensing system
Super-resolution

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Enhancing the resolution of a sensing system

Object of interest $x(t)$

Observed signal $y(t) = (x \ast h)(t)$

$\hat{x}(\omega)$ Fourier transform of $x(t)$

$\hat{y}(\omega) = \hat{x}(\omega)\hat{h}(\omega)$
Outline

1. Continuous case
2. Discrete case
3. Discrete case with noisy data
4. Solver for Problem 1
weighted superposition of spikes

\[ x = \sum_{j} a_j \delta_{t_j} \quad t_j \in [0, 1], \, a_j \in \mathbb{C} \]  

\[ y(k) = \sum_{j} a_j e^{-i2\pi k t_j} \quad k \in \mathbb{Z}, \, |k| \leq f_c \]  

\[ y = F_n x \quad n = 2f_c + 1 \]  

resolution cut-off \( \lambda_c := 1/f_c \)
Continuous case in 1D: notations

Weighted superposition of spikes

\[ x = \sum_j a_j \delta_{t_j} \quad t_j \in [0, 1], \ a_j \in \mathbb{C} \]  

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resolution cut-off \( \lambda_c := 1/f_c \)

Problem 1

\[ \min_{\tilde{x}} \| \tilde{x} \|_{TV} \quad \text{subject to } F_n \tilde{x} = y \]
Continuous case: theorem

Theorem 1.2

If $\Delta(T) \geq 2/f_c = 2\lambda_c$, then $x$ is the unique solution of Problem 1.
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Continuous case: theorem

**Theorem 1.2**

If \( \Delta(T) \geq 2/f_c = 2\lambda_c \), then \( x \) is the unique solution of Problem 1.

**Higher dimensions**

Also holds with a constant \( c_d \) instead of 2. (2.38 in 2D)
Sufficient condition: dual polynomials

\[ \forall v \in \mathbb{C}^{|T|}, |v_j| = 1, \text{ there exists a low frequency trigonometric polynomial} \]

\[ q(t) = \sum_{k=-f_c}^{f_c} c_k e^{i2\pi kt} \quad \text{s.t.} \quad \begin{cases} q(t_j) = v_j, & t_j \in T, \\ |q(t)| < 1, & t \in [0,1) \setminus T \end{cases} \]  \hspace{1cm} (5)

Analog to the sufficient condition for discrete signal in compressed sensing
Ideas of proof (2)

### Building $q$

$$q(t) = \sum_{t_j \in T} \alpha_j K(t - t_j) + \beta_j K'(t - t_j) \quad (6)$$

s.t.

$$\forall t_k \in T, \begin{cases} q(t_k) = v_k \\ q'(t_k) = 0 \end{cases} \quad (7)$$

### Squared Fejer kernel

$$K(t) = \left[ \frac{\sin \left( (\frac{f_c}{2} + 1) \pi t \right)}{\left( \frac{f_c}{2} + 1 \right) \sin(\pi t)} \right]^4 \quad (8)$$

$K$ and its derivatives decay rapidly around the origin.
Generalization to splines

Spline of order \( l \) in \( C^{l-1} \)

\[
x(t) = \sum_{t_j \in T} 1_{[t_{j-1}, t_j]} p_j(t) \quad \text{of period 1}
\] (9)
Generalization to splines

**Spline of order \( l \) in \( C^{l-1} \)**

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\]

of period 1 \((9)\)

**Recovery**

\[
x^{(l+1)} = \sum_j (p^{(l)}_{j+1}(t_j) - p^{(l)}_j(t_j)) \delta_{t_j}
\]

\((10)\)

\[
y^{(l+1)}_k = (i2\pi k)^{l+1} y_k, \quad k \neq 0
\]

\((11)\)

Periodicity \(\Rightarrow\)

\[
y^{(j)}_0 = \int_0^1 x^{(j)}(t) dt = 0, \quad 1 \leq j \leq l + 1
\]

\((12)\)
Discrete case

\[ x \in \mathbb{C}^N \]

\[ y_k = \sum_{t=0}^{N-1} x_t e^{-i2\pi kt/N} \quad |k| \leq f_c \]  \hspace{1cm} (13)
Discrete case

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**Corollary 1.4**

Let \( T \subset \{0, 1, \ldots, N-1\} \) be the support of \( \{x_t\}_{t=0}^{N-1} \) with

\[ \min_{t \neq t' \in T} |t - t'|/N \geq 2\lambda_c = 2/f_c. \] (14)

\( x \) is unique solution of

\[ \min_{\tilde{x}} \|\tilde{x}\|_1 \quad \text{subject to} \quad F_n\tilde{x} = y \] (Problem 2) (15)
Discrete case

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**Super-Resolution Factor** \( SRF = \frac{N}{n} \approx \frac{N}{2f_c} \)

If non zeros components of \( x \) are separated by at least \( 4SRF \), perfect super-resolution occurs.
Discrete case with noisy data

### Noisy data

<table>
<thead>
<tr>
<th>Equation</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y = F_n x + w$</td>
<td>$|F_n^*w|_1/N \leq \delta$ (16)</td>
</tr>
<tr>
<td>$y = F_n(x + z)$</td>
<td>$|z|_1 \leq \delta$, $z = P_n z$ (17)</td>
</tr>
<tr>
<td>$s := F_n^*y/N = P_n x + P_n z$</td>
<td>$|P_n z|_1 \leq \delta$ (18)</td>
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</table>

**Theorem 1.5**

Solution of Problem 3, under separation condition, obeys

$$\|\tilde{x} - x\|_1 \leq C_0 S R F^2 \delta$$ (20)
Discrete case with noisy data

Noisy data

\[ y = F_n x + w \quad \|F_n^* w\|_1 / N \leq \delta \]  (16)

\[ y = F_n(x + z) \quad \|z\|_1 \leq \delta, z = P_n z \]  (17)

\[ s := F_n^* y / N = P_n x + P_n z \quad \|P_n z\|_1 \leq \delta \]  (18)

Problem 3: relaxed version of Problem 2

\[ \min_{\tilde{x}} \|\tilde{x}\|_1 \quad \text{subject to} \quad \|P_n \tilde{x} - s\|_1 \leq \delta \]  (19)
Discrete case with noisy data

**Noisy data**

\[ y = F_n x + w \quad \|F_n^* w\|_1 / N \leq \delta \]  
\[ y = F_n(x + z) \quad \|z\|_1 \leq \delta, \ z = P_n z \]  
\[ s := F_n^* y / N = P_n x + P_n z \quad \|P_n z\|_1 \leq \delta \]

**Problem 3: relaxed version of Problem 2**

\[ \min_{\tilde{x}} \|\tilde{x}\|_1 \quad \text{subject to} \quad \|P_n \tilde{x} - s\|_1 \leq \delta \]

**Theorem 1.5**

Solution of Problem 3, under separation condition, obeys

\[ \|\tilde{x} - x\|_1 \leq C_0 SRF^2 \delta \]
Decompose the error into low and high frequency:

**Low frequency**

\[ \| h_L \|_1 = \| P_n (\hat{x} - x) \|_1 \leq \| P_n \hat{x} - s \|_1 + \| s - P_n x \|_1 \leq 2\delta \]  

**High frequency**

- \( \hat{x} \) as minimum \( l_1 \)-norm
- \( F_n h = 0 \Rightarrow \| P_T h \|_1 \leq (1 - \alpha/SRF^2) \| P_{T^c} h \|_1 \)
Problem 1

$$\min_{\tilde{x}} \|\tilde{x}\|_{TV} \quad \text{subject to } F_n \tilde{x} = y$$ (22)

Problem 1 (dual)

$$\max c^T \Re < y, c > \quad \text{subject to } \|F^* c\|_\infty \leq 1 \quad (23)$$

Corollary 4.1

$$\|F^* c\|_\infty \leq 1 \iff \exists Q \in \mathbb{C}^{n \times n}, (Q c c^*) \succeq 0, n - j \sum_{i=1}^{n} = \begin{cases} 1, & j = 0 \\ 0, & j \neq 0 \end{cases}$$ (24)
Problem 1

\[
\begin{align*}
\min_{\tilde{x}} & \quad \|\tilde{x}\|_{TV} \\
\text{subject to} & \quad F_n\tilde{x} = y
\end{align*}
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\|F^*_n c\|_\infty \leq 1 \iff \exists Q \in C_{n \times n}, \quad (Q c c^*_1) \succeq 0, \quad n - j \sum_{i=1}^{n-1} = \begin{cases} 1 & j = 0 \\ 0 & j \neq 0 \end{cases}
\] (24)
Solver for Problem 1

Problem 1

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\min_{\tilde{x}} \|\tilde{x}\|_{TV} \quad \text{subject to } F_n\tilde{x} = y
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(24)

Equivalent of the dual of problem 1

\[ \max_{c,Q} Re < y, c > \quad \text{subject to} \ (24) \]  

(25)
Conclusion

- Under minimum separation condition, super-resolution occurs.
- With noise, signal can be recovered with error proportional to noise, and to the square of the super-resolution factor

Thanks for your attention