Emmanuel Candes Reading Group
Towards a Mathematical Theory of Super-Resolution
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Super-resolution

Goal

Enhancing the resolution of a sensing system

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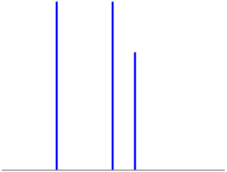
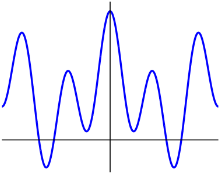
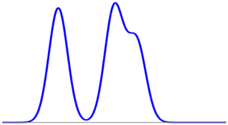
object of interest $x(t)$

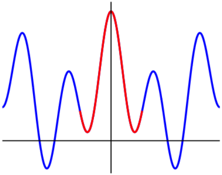




$\hat{x}(\omega)$ Fourier transform of $x(t)$

Observed signal $y(t) = (x \star h)(t)$





$\hat{y}(\omega) = \hat{x}(\omega)\hat{h}(\omega)$

Outline

Continuous case in 1D: notations

x Weighted superposition of spikes

$$x = \sum_j a_j \delta_{t_j} \qquad t_j \in [0, 1], a_j \in \mathbb{C} \tag{1}$$

$$y(k) = \sum_j a_j e^{-i2\pi k t_j} \qquad k \in \mathbb{Z}, |k| \leqslant f_c \tag{2}$$

$$y = F_n x \qquad n = 2f_c + 1 \tag{3}$$

resolution cut-off $\lambda_c := 1/f_c$

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Problem 1

$$\min_{\tilde{x}} \|\tilde{x}\|_{TV} \qquad \text{subject to } F_n \tilde{x} = y \tag{4}$$

Continuous case: theorem

Theorem 1.2

If $\Delta(T) \geqslant 2/f_c = 2\lambda_c$, then x is the unique solution of Problem 1.

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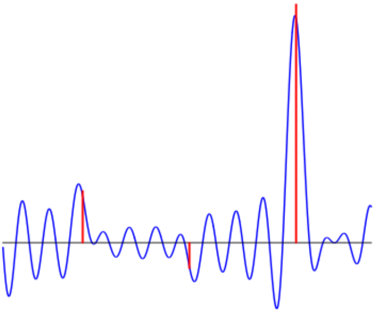

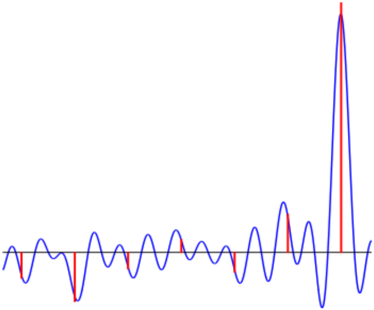

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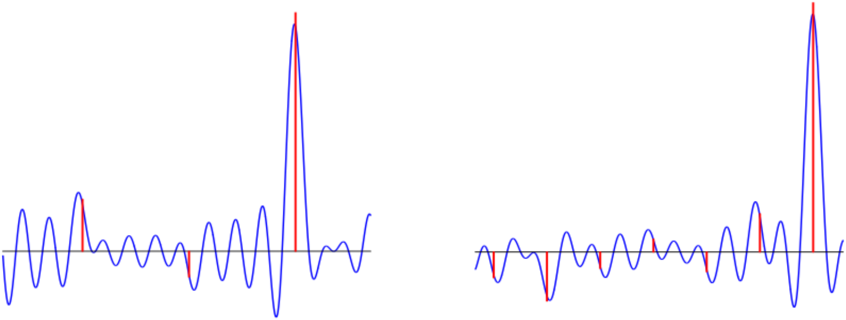

Higher dimensions

Also holds with a constant c_d instead of 2. (2.38 in 2D)

Ideas of proof (1)

Sufficient condition: dual polynomials

$\forall v \in \mathbb{C}^{|T|}, |v_j| = 1$, there exists a low frequency trigonometric polynomial

$$q(t) = \sum_{k=-f_c}^{f_c} c_k e^{i2\pi kt} \quad \text{s.t.} \quad \begin{cases} q(t_j) = v_j, & t_j \in T, \\ |q(t)| < 1, & t \in [0,1) \setminus T \end{cases} \tag{5}$$

Analog to the sufficient condition for discrete signal in compressed sensing

Ideas of proof (2)

Building q

$$q(t) = \sum_{t_j \in T} \alpha_j K(t - t_j) + \beta_j K'(t - t_j) \tag{6}$$

s.t.

$$\forall t_k \in T, \quad \begin{cases} q(t_k) = v_k \\ q'(t_k) = 0 \end{cases} \tag{7}$$

Squared Fejer kernel

$$K(t) = \left[\frac{sin\left(\left(\frac{f_c}{2} + 1 \right) \pi t \right)}{\left(\frac{f_c}{2} + 1 \right) sin(\pi t)} \right]^4 \tag{8}$$

K and its derivatives decay rapidly around the origin

Generalization to splines

Spline of order l in C^{l-1}

$$x(t) = \sum_{t_j \in T} \mathbf{1}_{[t_{j-1}, t_j]} p_j(t) \qquad \text{of period } 1 \tag{9}$$

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Recovery

$$x^{(l+1)} = \sum_j (p_{j+1}^{(l)}(t_j) - p_j^{(l)}(t_j))\delta_{t_j} \tag{10}$$

$$y_k^{(l+1)} = (i2\pi k)^{l+1} y_k, \qquad k \neq 0 \tag{11}$$

$$\text{periodicity} \Rightarrow y_0^{(j)} = \int_0^1 x^{(j)}(t)dt = 0, \qquad 1 \leqslant j \leqslant l+1 \tag{12}$$

Discrete case

$x \in \mathbb{C}^N$

$$y_k = \sum_{t=0}^{N-1} x_t e^{-i2\pi kt/N} \qquad |k| \leqslant f_c \tag{13}$$

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Corollary 1.4

Let $T \subset \{0, 1, ..., N-1\}$ be the support of $\{x_t\}_{t=0}^{N-1}$ with

$$\min_{t \neq t' \in T} |t - t'|/N \geqslant 2\lambda_c = 2/f_c. \tag{14}$$

x is unique solution of

$$\min_{\tilde{x}} \|\tilde{x}\|_1 \qquad \text{subject to } F_n\tilde{x} = y \qquad \text{(Problem 2)} \tag{15}$$

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Super-Resolution Factor $SRF = N/n \approx N/2f_c$

If non zeros components of x are separated by at least $4SRF$, perfect super-resolution occurs.

Discrete case with noisy data

Noisy data

$$y = F_n x + w \qquad \|F_n^* w\|_1 / N \leqslant \delta \tag{16}$$

$$y = F_n(x + z) \qquad \|z\|_1 \leqslant \delta, z = P_n z \tag{17}$$

$$s := F_n^* y / N = P_n x + P_n z \qquad \|P_n z\|_1 \leqslant \delta \tag{18}$$

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Problem 3: relaxed version of Problem 2

$$\min_{\tilde{x}} \|\tilde{x}\|_1 \qquad \text{subject to } \|P_n \tilde{x} - s\|_1 \leqslant \delta \tag{19}$$

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Theorem 1.5

Solution of Problem 3, under separation condition, obeys

$$\|\tilde{x} - x\|_1 \leqslant C_0 SRF^2 \delta \tag{20}$$

Ideas of proof

Decompose the error into low and high frequency:

Low frequency

$$\|h_L\|_1 = \|P_n(\hat{x} - x)\|_1 \leqslant \|P_n\hat{x} - s\|_1 + \|s - P_n x\|_1 \leqslant 2\delta \qquad (21)$$

High frequency

- $\tilde{x}$ as minimum l1-norm
- $F_n h = 0 \Rightarrow \|P_T h\|_1 \leqslant (1 - \alpha/SRF^2)\|P_{T^c} h\|_1$

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Corollary 4.1

$$\|F_n^* c\|_\infty \leqslant 1 \Leftrightarrow \exists Q \in \mathbb{C}^{n \times n}, \begin{pmatrix} Q & c \\ c* & 1 \end{pmatrix} \succcurlyeq 0, \sum_{i=1}^{n-j} = \begin{cases} 1, j = 0 \\ 0, j \neq 0 \end{cases} \qquad (24)$$

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Equivalent of the dual of problem 1

$$max_{c,Q} Re < y, c > \qquad \text{subject to (24)} \tag{25}$$

Conclusion

- Under minimum separation condition, super-resolution occurs.
- With noise, signal can be recovered with error proportional to noise, and to the square of the super-resolution factor

Thanks for your attention