TEXTURE MODELING
BY GAUSSIAN FIELDS WITH
PRESCRIBED LOCAL ORIENTATION

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joint work with

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Outline

- Introduction
  - Motivation
  - General probabilistic framework
- Our new stochastic model
  - Definition: Locally Anisotropic Fractional Brownian Field
  - Notion of tangent field
- Synthesis methods
  - Tangent field simulation by a turning bands method
  - LAFBF simulation via tangent field formulation
- Numerical experiments
- Conclusion and future work
How to synthesize natural random textures?

Mathematical model?
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Randomness
Self-similarity

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Roughness and regularity
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Roughness and regularity

Polisano et al. - Texture modeling by Gaussian field with prescribed local orientation
How to synthesize natural random textures?

- Randomness
- Self-similarity
- Roughness and regularity
- Mathematical model?
How to synthesize natural random textures?

Mathematical model?

Randomness
Self-similarity

Orientation and anisotropy

Roughness and regularity

Polisano et al. - Texture modeling by Gaussian field with prescribed local orientation
The basic component:

**Fractional Brownian Field (FBF)**

- $B^H$ FBF with Hurst index $0 < H < 1$ [Mandelbrot, Van Ness, 1968]
- stationary increments: $B^H(\cdot + z) - B^H(z) \overset{\mathcal{L}}{=} B^H(\cdot) - B^H(0)$
- self-similar: $B^H(\lambda \cdot) \overset{\mathcal{L}}{=} \lambda^H B^H(\cdot)$
- isotropic: $B^H \circ R_\theta \overset{\mathcal{L}}{=} B^H$

The covariance is given by

$$\text{Cov}(B^H(x), B^H(y)) = c_H(\|x\|^{2H} + \|y\|^{2H} - \|x - y\|^{2H})$$
The basic component:
Fractional Brownian Field (FBF)

Harmonizable representation

\[
B^H(x) = \int_{\mathbb{R}^2} \frac{e^{ix \cdot \xi} - 1}{\| \xi \|^{H+1}} d\hat{\mathcal{W}}(\xi)
\]

[Samorodnitsky, Taqqu, 1997]
The basic component: Fractional Brownian Field (FBF)

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The basic component: Fractional Brownian Field (FBF)

Harmonizable representation

\[ B^H(x) = \int_{\mathbb{R}^2} \frac{e^{ix \cdot \xi} - 1}{\|\xi\|_H^{H+1}} d\widehat{W}(\xi) \]

roughness indicator

complex Brownian measure

[Samorodnitsky, Taqqu, 1997]
The basic component:

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roughness indicator
complex Brownian measure

H=0.2

[Samorodnitsky, Taqqu, 1997]
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Fractional Brownian Field (FBF)

- Harmonizable representation

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- Roughness indicator
- Complex Brownian measure

[Samorodnitsky, Taqqu, 1997]
General model:

anisotropic self-similar Gaussian fields

\[ X(x) = \int_{\mathbb{R}^2} (e^{ix \cdot \xi} - 1) f^{1/2}(x, \xi) d\widehat{W}(\xi) \]
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f^{1/2}(x, \xi) = c(x, \xi) ||\xi||^{-h(x, \xi)} - 1
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X(x) = \int_{\mathbb{R}^2} (e^{ix \cdot \xi} - 1) f^{1/2}(x, \xi) d\tilde{W}(\xi)
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- \( c(x, \xi) \equiv 1 \) and \( h(x, \xi) \equiv H \) \( \Rightarrow X = B^H \)  

[Mandelbrot, Van Ness, 1968]
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- \( c(x, \xi) \equiv c(\text{arg} \, \xi) \) and \( h(x, \xi) \equiv h(\text{arg} \, \xi) \) \( \Rightarrow X = AFBF \) [Bonami, Estrade, 2003]

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- **Example : elementary fields** \( c(\arg \xi) = 1_{[-\alpha, \alpha]}(\arg \xi - \alpha_0) \)
  
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- \(c(x, \xi) \equiv c(\arg \xi)\) and \(h(x, \xi) \equiv h(\arg \xi) \Rightarrow X = AFB\) \([\text{Bonami, Estrade, 2003}]\)

**Example:** *elementary fields* \(c(\arg \xi) = \mathbb{1}_{[-\alpha, \alpha]}(\arg \xi - \alpha_0)\)  
\([\text{Bierme, Richard, Moisan, 2012}]\)
General model: anisotropic self-similar Gaussian fields

\[ X(x) = \int_{\mathbb{R}^2} (e^{ix \cdot \xi} - 1) f^{1/2}(x, \xi) d\hat{W}(\xi) \]

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[Polisano, Clausel, Perrier, Condat, 2014]
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General model:
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Global Anisotropy

Local orientation

Local roughness

[Mandelbrot, Van Ness, 1968]
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- \[ c(x, \xi) \equiv 1 \text{ and } h(x, \xi) \equiv H \Rightarrow X = B^H \] \[ \text{[Mandelbrot, Van Ness, 1968]} \]
- \[ c(x, \xi) \equiv c(\arg \xi) \text{ and } h(x, \xi) \equiv h(\arg \xi) \Rightarrow X = AFBF \] \[ \text{[Bonami, Estrade, 2003]} \]
- Example: \textit{elementary fields} \[ c(\arg \xi) = 1_{[-\alpha, \alpha]}(\arg \xi - \alpha_0) \] \[ \text{[Bierme, Richard, Moisan, 2012]} \]
- \[ c(x, \xi) \equiv c(x, \arg \xi) \text{ and } h(x, \xi) \equiv h(x) \] \[ \text{[Polisano, Clausel, Perrier, Condat, 2014]} \]

Local roughness
General model:

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[Mandelbrot, Van Ness, 1968]
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**Definition**: Our new Gaussian model LAFBF is a local version of the elementary field

\[
B_{\alpha_0,\alpha}^H(x) = \int_{\mathbb{R}^2} (e^{ix \cdot \xi} - 1) \frac{1_{[-\alpha,\alpha]}(\arg \xi - \alpha_0(x))}{\|\xi\|^{H+1}} d\hat{W}(\xi)
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[Polisano et al., 2014]

The orientation may vary spatially. \(\alpha_0\) is now a differentiable function on \(\mathbb{R}^2\).
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The orientation may vary spatially. \(\alpha_0\) is now a differentiable function on \(\mathbb{R}^2\)

[Polisano et al., 2014]
Elementary field

\[ \alpha_0 = 0 \]

\[ \alpha = -\frac{\pi}{2} \]
Elementary field

\[ \alpha_0 = 0 \]

\[ \alpha = 0.7 \]
Elementary field

\[ \alpha_0 = 0 \]

\[ \alpha = 0.6 \]
Elementary field

\[ \alpha_0 = 0 \]

\[ \alpha = 0.5 \]
Elementary field

\[ \alpha_0 = 0 \]

\[ \alpha = 0.4 \]
Elementary field

\[ \alpha_0 = 0 \]

\[ \alpha = 0.3 \]
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**Elementary field**

\[ \alpha_0 = 0 \]

\[ \alpha = 0.2 \]
Elementary field

\[ \alpha_0 = 0 \]

\[ \alpha = 0.1 \]
Elementary field

\[ \alpha_0 = 0 \]

\[ \alpha = 0.05 \]
Elementary field

\[ \alpha_0 = -\frac{\pi}{3} \]

\[ \alpha = 0.7 \]
Elementary field

\[ \alpha_0 = -\frac{\pi}{3} \]

\[ \alpha = 0.6 \]
Elementary field

\[ \alpha_0 = -\frac{\pi}{3} \]

\[ \alpha = 0.5 \]

Texture orientation

\[ \mathbf{V}_{x_0} \]
Elementary field

\[ \alpha_0 = -\frac{\pi}{3} \]

\[ \alpha = 0.4 \]
Elementary field

\[ \alpha_0 = -\frac{\pi}{3} \]

\[ \alpha = 0.3 \]
Elementary field

\[ \alpha_0 = -\frac{\pi}{3} \]

\[ \alpha = 0.2 \]
Elementary field

\[ \alpha_0 = -\frac{\pi}{3} \]

\[ \alpha = 0.1 \]
Elementary field

\[ \alpha_0 = -\frac{\pi}{3} \]

\[ \alpha = 0.05 \]
Tangent field

For a random field $X$ locally asymptotically self-similar of order $H$, 

$$B_{\alpha_0,\alpha}^H(x) = \int_{\mathbb{R}^2} (e^{ix \cdot \xi} - 1) \frac{\mathbb{1}_{[-\alpha,\alpha]}(\arg \xi - \alpha_0(x))}{\|\xi\|^{H+1}} d\hat{W}(\xi)$$

[Tangent field.]

$Y_{x_0}$ : tangent field of $X$ at point $x_0 \in \mathbb{R}^2$

Deterministic case

**Taylor’s expansion**  ↔  **Tangent field**

Stochastic case

[Benassi, 1997]

[Falconer, 2002]
Tangent field

For a random field $X$ locally asymptotically self-similar of order $H$,

$$B^H_{\alpha_0,\alpha}(x) = \int_{\mathbb{R}^2} (e^{ix \cdot \xi} - 1) \frac{1_{[-\alpha,\alpha]}(\arg \xi - \alpha_0(x))}{\|\xi\|^{H+1}} d\hat{W}(\xi)$$

$Y_{x_0}$: tangent field of $X$ at point $x_0 \in \mathbb{R}^2$  

[Benassi, 1997]  
[Falconer, 2002]
**Theorem.** The LAFBF $B^H_{\alpha_0, \alpha}$ admits for tangent field $Y_{x_0}$:

$$Y_{x_0}(x) = \int_{\mathbb{R}^2} (e^{ix \cdot \xi} - 1) \frac{1_{[-\alpha, \alpha]}(\arg \xi - \alpha_0(x))}{\| \xi \|^{H+1}} d\hat{W}(\xi)$$

$Y_{x_0}$ elementary field with global orientation $\alpha_0(x_0)$
Tangent field

\[
B^H_{\alpha_0, \alpha}(x) = \int_{\mathbb{R}^2} \left( e^{ix \cdot \xi} - 1 \right) \frac{1_{[-\alpha, \alpha]}(\arg \xi - \alpha_0(x))}{\|\xi\|^{H+1}} d\tilde{W}(\xi)
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\textbf{Theorem.} The LAFBF \( B^H_{\alpha_0, \alpha} \) admits for tangent field \( Y_{x_0} \):

\[
Y_{x_0}(x) = \int_{\mathbb{R}^2} \left( e^{ix \cdot \xi} - 1 \right) \frac{1_{[-\alpha, \alpha]}(\arg \xi - \alpha_0(x_0))}{\|\xi\|^{H+1}} d\tilde{W}(\xi)
\]

\( \rightarrow Y_{x_0} \) \textit{elementary field} with global orientation \( \alpha_0(x_0) \)
Tangent field

\[ B^{H}_{\alpha_0, \alpha}(x) = \int_{\mathbb{R}^2} (e^{ix \cdot \xi} - 1) \frac{1[\arg \xi - \alpha_0(x)]}{\|\xi\|_{H+1}} d\hat{W}(\xi) \]

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\[ \rightarrow Y_{x_0} \text{ elementary field with global orientation } \alpha_0(x_0) \]

\[ B^{H}_{\alpha_0, \alpha}(x_0) \approx Y_{x_0}(x = x_0) \]
Simulation of tangent fields

**Continuous formulation.** Variogram of $Y_{x_0}$: [Bierme, Richard, Moisan, 2012]

$$
\nu_{Y_{x_0}}(x) = \frac{1}{2} \int_{\mathbb{R}^2} |e^{ix \cdot \xi} - 1|^2 f(x_0, \xi) d\xi
$$

$$
= \frac{1}{2} \gamma(H) \int_{-\pi/2}^{\pi/2} c_{\alpha_0, \alpha}(x_0, \theta) |x \cdot u(\theta)|^{2H} d\theta
$$

$$
= \int_{-\pi/2}^{\pi/2} \tilde{\nu}_\theta(x \cdot u(\theta)) d\theta
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Simulation of tangent fields

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Simulation of tangent fields

Continuous formulation. Variogram of $Y_{x_0}$: [Bierme, Richard, Moisan, 2012]

$$v_{Y_{x_0}}(x) = \frac{1}{2} \int_{\mathbb{R}^2} |e^{ix \cdot \xi} - 1|^2 f(x_0, \xi) d\xi$$

in polar coordinates

$$= \frac{1}{2} \gamma(H) \int_{-\pi/2}^{\pi/2} c_{\alpha_0, \alpha}(x_0, \theta) |x \cdot u(\theta)|^{2H} d\theta$$

$$= \int_{-\pi/2}^{\pi/2} \tilde{v}_\theta(x \cdot u(\theta)) d\theta$$

$$\tilde{v}_\theta = \frac{1}{2} \gamma(H) c_{\alpha_0, \alpha}(x_0, \theta) \cdot |x|^{2H}$$

$$u(\theta) = (\cos \theta, \sin \theta)$$

$$\gamma(H) = \frac{\pi}{H \Gamma(2H) \sin(H\pi)}$$
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variogram of a fractional brownian motion (FBM) of order $H$

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variogram of a fractional brownian motion (FBM) of order $H$

$Y_{x_0}$

= Infinite sum of independent rotating FBM of order $H$

$$\tilde{v}_\theta = \frac{1}{2} \gamma(H) c_{\alpha_0, \alpha}(x_0, \theta) \cdot |^2H$$

$u(\theta) = (\cos \theta, \sin \theta)$

$$\gamma(H) = \frac{\pi}{H \Gamma(2H) \sin(H\pi)}$$
Simulation of tangent fields

Discrete formulation. [Bierme, Richard, Moisan, 2012]

- $(\theta_i)_{1 \leq i \leq n}$ are $n$ bands orientations and $\lambda_i = \theta_{i+1} - \theta_i$

- The turning band field is defined as

$$Y_{x_0}^{[n]}(x) = \gamma(H)^{\frac{1}{2}} \sum_{i=1}^{n} \sqrt{\lambda_i} c_{\alpha_0, \alpha}(x_0, \theta_i) B_i^H(x \cdot u(\theta_i))$$

- $B_i^H$ are $n$ independent FBM of order $H$

- Good approximation provided $\max_i \lambda_i \leq \varepsilon$
Simulation of tangent fields

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$$Y_{x_0}[n](x) = \gamma(H)^{\frac{1}{2}} \sum_{i=1}^{n} \sqrt{\lambda_i c_{\alpha_0,\alpha}(x_0, \theta_i)} B_i^H (x \cdot u(\theta_i))$$

- $B_i^H$ are $n$ independent FBM of order $H$

- Good approximation provided $\max_i \lambda_i \leq \varepsilon$

[Bierme, Richard, Moisan, 2012]
Simulation of tangent fields

Simulation along particular bands.  

Discrete grid $r^{-1}\mathbb{Z}^2 \cap [0, 1]^2$ with $r = 2^k - 1, \ k \in \mathbb{N}^*$

Choose $(\theta_i)$ such that $\tan \theta_i = \frac{p_i}{q_i}$ and $\max \lambda_i \leq \epsilon$

Then $B_i^H(x \cdot u(\theta_i))$ becomes

$$\left\{ B_i^H \left( \frac{k_1}{r} \cos \theta_i + \frac{k_2}{r} \sin \theta_i \right) ; 0 \leq k_1, k_2 \leq r \right\} \stackrel{\mathcal{L}}{=} \left\{ B_i^H \left( \frac{\cos \theta_i}{rq_i} \right)^H (k_1 q_i + k_2 p_i) ; 0 \leq k_1, k_2 \leq r \right\}$$
Simulation of tangent fields

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\[
\left\{ B_i^H \left( \frac{k_1}{r} \cos \theta_i + \frac{k_2}{r} \sin \theta_i \right) ; 0 \leq k_1, k_2 \leq r \right\} \overset{\mathcal{L}}{=} \left\{ \left( \frac{\cos \theta_i}{rq_i} \right)^H (k_1 q_i + k_2 p_i) ; 0 \leq k_1, k_2 \leq r \right\}
\]
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\[
\left\{ \begin{array}{l}
B_i^H \left( \frac{k_1}{r} \cos \theta_i + \frac{k_2}{r} \sin \theta_i \right); 0 \leq k_1, k_2 \leq r \\
\left( \frac{\cos \theta_i}{rq_i} \right)^H \{ B_i^H (k_1 q_i + k_2 p_i); 0 \leq k_1, k_2 \leq r \}
\end{array} \right.
\]

Dynamic programming

Polisano et al. - Texture modeling by Gaussian field with prescribed local orientation
Simulation of tangent fields

Simulation along particular bands.

- Discrete grid $r^{-1} \mathbb{Z}^2 \cap [0, 1]^2$ with $r = 2^k - 1, k \in \mathbb{N}^*$

- Choose $\{\theta_i\}$ such that $\tan \theta_i = \frac{p_i}{q_i}$ and $\max_i \lambda_i \leq \epsilon$

- Then $B^H_i(x \cdot u(\theta_i))$ becomes

$$
\left\{ B^H_i \left( \frac{k_1}{r} \cos \theta_i + \frac{k_2}{r} \sin \theta_i \right) ; 0 \leq k_1, k_2 \leq r \right\} \triangleq \left\{ B^H_i \left( \frac{\cos \theta_i}{rq_i} \right)^H \left( \frac{\cos \theta_i}{rq_i} \right) \right\} \triangleq \left\{ B^H_i \left( k_1 q_i + k_2 p_i \right) ; 0 \leq k_1, k_2 \leq r \right\}
$$

Dynamic programming

Equispaced

Self-similarity

$B^H(\lambda; \cdot) \triangleq \lambda^H B^H(\cdot)$

Dynamic programming

Equispaced

[Polisano et al. - Texture modeling by Gaussian field with prescribed local orientation]
Simulation of LAFBF using tangent fields

Algorithm. For each pixel $x_0 = (k_1, k_2) \in [0, r]^2$

$$B^H_{\alpha_0, \alpha}(k_1, k_2) = \gamma(H)^{1/2} \sum_{i=1}^{n} \sqrt{\lambda_i} c_{\alpha_0, \alpha}(k_1, k_2, \theta_i) \left( \frac{\cos \theta_i}{r q_i} \right)^H B^H_i (k_1 q_i + k_2 p_i)$$

[Polisano et al., 2014]
Simulation of LAFBF using tangent fields [Polisano et al., 2014]

**Algorithm.** For each pixel \( x_0 = (k_1, k_2) \in [0, r]^2 \)

\[
B^H_{\alpha_0, \alpha}((k_1, k_2)) \quad B^H_{\alpha_0, \alpha}(x_0) \approx Y_{x_0}(x = x_0) = \\
\gamma(H)^{1/2} \sum_{i=1}^{n} \sqrt{\lambda_i c_{\alpha_0, \alpha}((k_1, k_2), \theta_i)} \left( \frac{\cos \theta_i}{r q_i} \right)^H B^H_i(k_1 q_i + k_2 p_i)
\]
Algorithm. For each pixel $x_0 = (k_1, k_2) \in [0, r]^2$

$$\mathbf{B}_{\alpha_0, \alpha}^H((k_1, k_2))$$

$$= \gamma(H)^{\frac{1}{2}} \sum_{i=1}^{n} \sqrt{\lambda_i c_{\alpha_0, \alpha}((k_1, k_2), \theta_i)} \left( \frac{\cos \theta_i}{r q_i} \right)^H B_i^H (k_1 q_i + k_2 p_i)$$

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\]
Algorithm. For each pixel $x_0 = (k_1, k_2) \in [0, r]^2$

$$B^H_{\alpha_0, \alpha}((k_1, k_2))$$

$$= \gamma(H) \frac{1}{2} \sum_{i=1}^{n} \sqrt{\lambda_i c_{\alpha_0, \alpha}((k_1, k_2), \theta_i)} \left( \frac{\cos \theta_i}{rq_i} \right)^H B^H_i(k_1q_i + k_2p_i)$$

$n$ turning bands $\theta_i$
Simulation of LAFBF using tangent fields

**Algorithm.** For each pixel $x_0 = (k_1, k_2) \in [0, r]^2$

$$B_H^{\alpha_0, \alpha}((k_1, k_2))$$

$$= \gamma(H)^{1/2} \sum_{i=1}^{n} \sqrt{\lambda_i c_{\alpha_0, \alpha}((k_1, k_2), \theta_i)} \left( \frac{\cos \theta_i}{r q_i} \right)^H B_i^H (k_1 q_i + k_2 p_i)$$

$n$ turning bands $\theta_i$

$1_{[-\alpha, \alpha]}(\theta_i - \alpha_0((k_1, k_2))) \neq 0$

$$\iff \left| \theta_i - \alpha_0((k_1, k_2)) \right| \leq \alpha$$
Algorithm. For each pixel $x_0 = (k_1, k_2) \in [0, r]^2$

$$B^{H}_{\alpha_0, \alpha}((k_1, k_2))$$

$$= \gamma(H)^{\frac{1}{2}} \sum_{i=1}^{n} \sqrt{\lambda_i} c_{\alpha_0, \alpha}((k_1, k_2), \theta_i) \left( \frac{\cos \theta_i}{rq_i} \right)^H B^H_i (k_1 q_i + k_2 p_i)$$

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Simulation of LAFBF using tangent fields

**Algorithm.** For each pixel $x_0 = (k_1, k_2) \in [0, r]^2$

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- $n$ turning bands $\theta_i$
- Few bands in the cone

$$\mathbb{1}_{[-\alpha, \alpha]}(\theta_i - \alpha_0((k_1, k_2))) \neq 0 \quad \iff \quad |\theta_i - \alpha_0((k_1, k_2))| \leq \alpha$$
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Algorithm. For each pixel $x_0 = (k_1, k_2) \in [0, r]^2$

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$n$ turning bands $\theta_i$

Few bands in the cone

$\mathbb{1}_{[-\alpha, \alpha]}(\theta_i - \alpha_0((k_1, k_2))) \neq 0$ $\iff$ $|\theta_i - \alpha_0((k_1, k_2))| \leq \alpha$

Complexity $O(r^2 \log n)$
Polisano et al. - Texture modeling by Gaussian field with prescribed local orientation

$\vec{V}_1^{(x,y)} = (\cos(-\pi/2 + y), \sin(-\pi/2))$

Numerical experiments

Parameters

$r = 255 \quad H = 0.2$

$\alpha = 10^{-1} \quad \epsilon = 10^{-2}$
Texture modeling by Gaussian field with prescribed local orientation

\[ \tilde{V}_{x,y}^1 = (\cos(-\pi/2 + y), \sin(-\pi/2)) \]

Texture with prescribed local orientation at each point \( x_0 \) given by a vector field

\[ \tilde{V}_{x_0} = u(\alpha_0(x_0)) \]

Parameters

\[ r = 255 \quad H = 0.2 \]
\[ \alpha = 10^{-1} \quad \epsilon = 10^{-2} \]
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A zoom around a point \( x_0 \) shows that locally a LAFBF behaves as an elementary field

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Regularized version of the anisotropy function

\[ f^{1/2}(x_0, \xi) = \frac{c_{\alpha,0}(x_0, \arg \xi)}{||\xi||^{\alpha+1}} \]

Parameters

\[ r = 255 \quad H = 0.2 \]
\[ \alpha = 10^{-1} \quad \epsilon = 10^{-2} \]
Numerical experiments

\[
\tilde{V}^1_{(x,y)} = (\cos(-\pi/2 + y), \sin(-\pi/2))
\]

Texture with prescribed local orientation at each point \(x_0\) given by a vector field

\[
\tilde{V}_{x_0} = u(\alpha_0(x_0))
\]

A zoom around a point \(x_0\) shows that locally a LAFBF behaves as an elementary field

Regularized version of the anisotropy function

\[
f^{1/2}(x_0; \xi) = \frac{c_{\alpha,\alpha}(x_0, \arg \xi)}{||\xi||^{\alpha+1}}
\]

Parameters

\[
\begin{align*}
    r &= 255 & H &= 0.2 \\
    \alpha &= 10^{-1} & \epsilon &= 10^{-2}
\end{align*}
\]
Numerical experiments

\[ \vec{V}^2_{(x,y)} = (\cos(\cos(36xy)), \sin(\cos(36xy))) \]

\[ \vec{V}^3_{(x,y)} = \nabla F(x, y) \]

\[ F(x, y) = (4x - 2)e^{-(4x-2)^2-(4y-2)^2} \]
Numerical experiments

\[ \vec{V}^2_{(x,y)} = (\cos(\cos(36xy)), \sin(\cos(36xy))) \quad \vec{V}^3_{(x,y)} = \nabla F(x, y) \]

\[ F(x, y) = (4x - 2)e^{-(4x-2)^2-(4y-2)^2} \]
Numerical experiments

\[ \mathbf{V}^2_{(x,y)} = (\cos(\cos(36xy)), \sin(\cos(36xy))) \]

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Numerical experiments

\[ V^1_{(x,y)} = (\cos(-\pi/2 + y), \sin(-\pi/2)) \]

\[ H = 0.2 \quad H = 0.5 \]
Numerical experiments

$V_1^{(x,y)} = (\cos(-\pi/2 + y), \sin(-\pi/2))$

$H=0.2$  $H=0.5$
Numerical experiments

\[ \vec{V}^1_{(x,y)} = (\cos(-\pi/2 + y), \sin(-\pi/2)) \]

\( H = 0.2 \)  \hspace{1cm}  \( H = 0.5 \)
Conclusion and future work

Conclusion
Conclusion

Introduce a new stochastic model defined as a local version of an AFBF.
Conclusion

- Introduce a **new stochastic model**
  defined as a local version of an AFBF.
- Simulations based on **tangent field formulation**
  and the **turning bands method** produce textures with
  prescribed local orientations.
Introduction

Introduce a **new stochastic model** defined as a local version of an AFBF.

Simulations based on **tangent field formulation** and the **turning bands method** produce textures with prescribed local orientations.
Conclusion and future work

**Conclusion**
- Introduce a **new stochastic model** defined as a local version of an AFBF.
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**Future work**
- Extensions of our model include
  - Hurst index may **vary spatially**.
**Conclusion**

- Introduce a **new stochastic model** defined as a local version of an AFBF.
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- Extensions of our model include Hurst index may **vary spatially**.
Conclusion and future work

**Conclusion**
- Introduce a *new stochastic model* defined as a local version of an AFBF.
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**Future work**
- Extensions of our model include Hurst index may *vary spatially*. 
Selected papers


Questions ?
Thank you for your attention.
Dynamic programming. The choice of the bands orientation \((\theta_i)_{1 \leq i \leq n}\) is governed by the computational cost of the \(B_i^H\)'s within dynamic programming.

Let the error \(\epsilon\) fixed. Taking \(N = \lceil \frac{1}{\tan \epsilon} \rceil\) consider the following set:

\[
\mathcal{V}_N = \left\{ (p, q) \in \mathbb{Z}^2 / -N \leq p \leq N, 1 \leq q \leq N, p \wedge q = 1, -\frac{\pi}{2} < \arctan \left( \frac{p}{q} \right) < \frac{\pi}{2} \right\}
\]

The aim is to find \(n\) pairs in the set \(\mathcal{V}_N\) which minimize the following global cost:

\[
C'(\Theta) = \sum_{k=1}^{s} C'(r(|p_{i_k}| + q_{i_k}))
\]

where \(C'(\ell)\) is the cost of one FBM \(B_i^H\) in \(O(n \log n)\), under the constraint \(\max_i (\theta_{i+1} - \theta_i) \leq \epsilon\)