Simulation of Geophysical Flows.

Carlos Parés[1].

[1]Universidad de Málaga(Spain))

ESSIM 2013. U. Carlos III, Madrid

Outline

Edanya group

Edanya: Main Goals

Development of robust, reliable and low computational cost numerical tools for the simulation of geophysical flows and the prediction of emergency situations such as river floodings or oil spills, tsunamis, debris avalanches …

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Ingredients

- Mathematical models: based in Shallow-water equations.
- Numerical methods: High order finite volume schemes.
- HySEA: High Performance Cloud Computing software to simulate geophysical flows.

HySEA: interdisciplinary platform

- Models based on geophysical flows with applications in:
  - Physical Oceanography
  - Marine Geology and Ecology
  - Tsunami Research
  - Civil and Hydraulic Engineering

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Dambreak problem: Limonero Dam (close to Málaga (Spain))

- Resolution 5 m $\times$ 5 m.
- Number of cells: 1052224.
- Real simulated time: 20 min.
- Used scheme: Second order HLL or PVM-1U(S_L,S_R).
- Positivity of the water height is ensured.
- Graphics card: GeForce GTX 570. Speedup: 230 (1 Intel Core i7 920).

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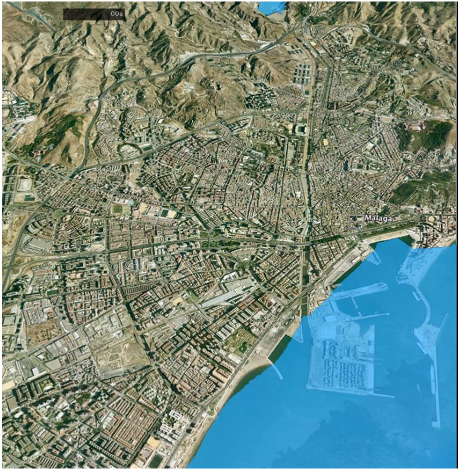
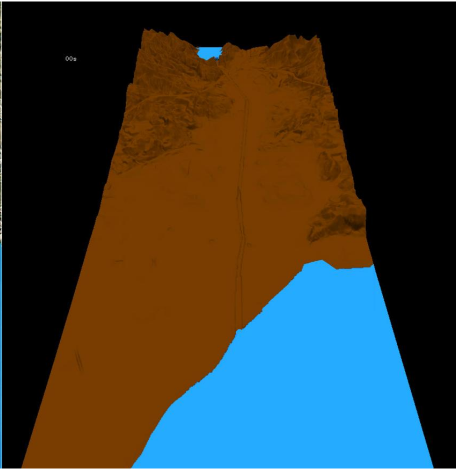
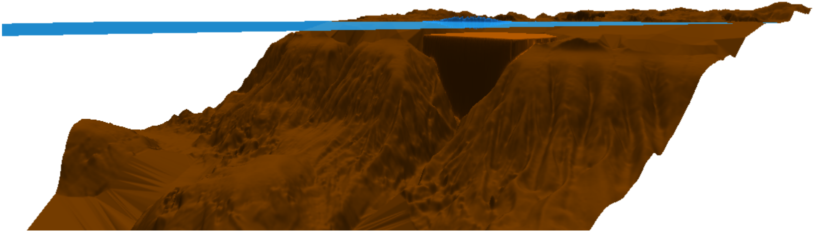




Tsunami generated by a submarine landslide in Sumatra

- Resolution 20 m $\times$ 20 m.
- Number of cells: 939500.
- Real simulated time: 12 min.
- Used scheme: Second order PVM-2U(S_L,S_R,S_int).

Tsunami generated by a submarine landslide in Sumatra



Shallow flow through a channel: notations

- Let us consider the free surface flow of a shallow layer of fluid through a channel with rectangular cross-section of constant width $2A$. Let L be the length.
- We consider a cartesian system of reference x, y, z such that the x axis is parallel the channel axis and $z = 0$ corresponds to the undisturbed free surface level.
- Let $H(x, y)$ be the depth of the channel measured from $z = 0$, i.e. the bottom is given by the surface Γ_b whose equation is:

$$z = -H(x, y) \qquad 0 \leq x \leq L, \quad -A \leq y \leq A.$$

- Let us denote by $\eta(x, y, t)$ the heigth of the free surface over the point $(x, y, 0)$ at time t. The free surface at time t is given by the surface Γ_s^t whose equation is:

$$z = \eta(x, y, t), \qquad 0 \leq x \leq L, \quad -A \leq y \leq A.$$

- Let us denote by $h(x, y, t)$ the thickness of the water layer over the point $(x, y, 0)$ at time t, i.e.

$$h(x, y, t) = \eta(x, y, t) + H(x, y), \quad \forall x \in [0, L].$$

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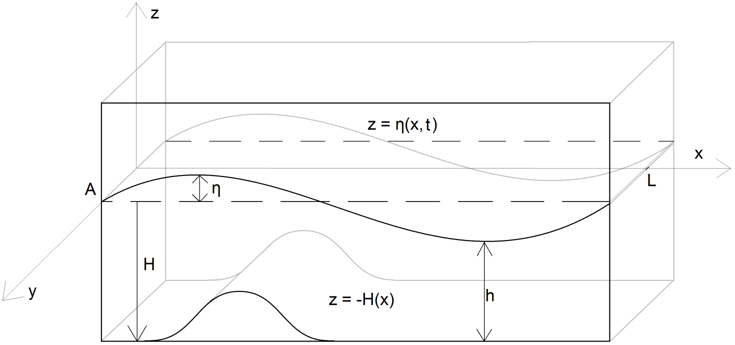

Shallow flow through a channel: notations

- Let us denote by Ω_t the 3D domain occupied by the water at time t:

$$\Omega_t = \left\{ (x, y, z) \in \mathbb{R}^3 \text{ s.t. } 0 \leq x \leq L, \quad -A \leq y \leq A, \quad -H(x, y) \leq z \leq \eta(x, y, t) \right\}.$$

- Let us define:

$$\mathcal{Q} = \left\{ (x, y, z, t) \in \mathbb{R}^3 \times \mathbb{R}^+ \text{ s.t. } (x, y, z) \in \Omega_t \right\}.$$

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Fundamental Hypotheses

- (FH1) Water is assumed to be a **continuous medium**, i.e.
  - Water particles are identified with the dimensionless points of Ω_t.
  - No mass can be assigned to the water particles, as there are uncountable infinitely many.
  - A mass $m(\mathcal{O}, t)$ should be assigned to the water occupying any measurable subset $\mathcal{O} \subset \Omega_t$.
  - We assume the existence of a locally integrable function $\rho : \mathcal{Q} \to \mathbb{R}^+$ such that

$$m(\mathcal{O}, t) = \int_{\mathcal{O}} \rho(x, y, z, t)\, dx\, dy\, dz,$$

    for any measurable subset $\mathcal{O}$ of Ω_t. The function ρ is called the **density** function.
  - We also assume the existence of a vector function $\vec{u} : \mathcal{Q} \mapsto \mathbb{R}^3$ such that the velocity of the water particle which is at the point (x, y, z) at time t is given by:

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    $\vec{u}$ is called the **velocity field**.
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$$\vec{q}(x, y, z, t) := h(x, y, t)\vec{u}(x, y, z, t)$$

    the **discharge**.

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$$\rho(x,y,z,t) = \rho_0, \quad \forall (x,y,z,t) \in Q.$$

  - As a consequence:

$$m(\mathcal{O},t) = \int_{\mathcal{O}} \rho(x,y,z,t)\, dx\, dy\, dz = \rho_0 \, |\mathcal{O}| \, ,$$

    where $|\mathcal{O}|$ represents the Lebesgue measure of $\mathcal{O}$.

- (FH3) Water flow is assumed to be **incompressible**, i.e. if $\mathcal{O}_t$ is the domain occupied at time t by the water particles that filled a subset $\mathcal{O}_0$ at time $t = 0$, then

$$|\mathcal{O}_t| = |\mathcal{O}_0| \, , \quad \forall t \in \mathbb{R}^+.$$

  - If $\vec{u}$ is assumed to be smooth enough, the incompressibility hypothesis is equivalent to assume that the divergence of $\vec{u}$ is zero:

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Fundamental Hypotheses

- (FH4) Pressure p is hydrostatic.
  - The pressure of a fluid at rest increases linearly with depth. More precisely, it satisfies the equation:
  $$\frac{\partial p}{\partial z} = -\rho g, \tag{1}$$
  where g is the gravity acceleration.
  - At the free surface, the pressure should be equal to the air pressure p_a:
  $$p(x, y, \eta(x, y, t)) = p_a(x, y, \eta(x, y, t)). \tag{2}$$
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  - In particular, **capilar waves** due to the surface tension are neglected.

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- (FH4) Pressure p is hydrostatic.
  - The pressure of a fluid at rest increases linearly with depth. More precisely, it satisfies the equation:

$$\frac{\partial p}{\partial z} = -\rho g, \tag{1}$$

    where g is the gravity acceleration.
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- (PH2) The depth function H only depends on the x variable:

$$H = H(x), \qquad 0 \leq x \leq L.$$

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Some calculations

- Given a and b such that $0 \leq a < b \leq L$ and $t > 0$, let us define

$$\mathcal{O}^t_{a,b} = \{(x,y,z) \quad \text{s.t} \quad a \leq x \leq b, \ -A \leq y \leq A, \ -H(x) \leq z \leq \eta(x,t)\}.$$

- The total mass of the water contained in $\mathcal{O}^t_{a,b}$ is:

$$\begin{aligned} m^t_{a,b} &= \rho_0 \int_a^b \int_{-H(x)}^{\eta(x,t)} \int_{-A}^{A} 1 \, dy \, dz \, dx \\ &= 2A\rho_0 \int_a^b (\eta(x,t) + H(x)) \, dx \\ &= 2A\rho_0 \int_a^b h(x,t) \, dx. \end{aligned}$$

where

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some easy computation show that the surface element is given by:

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Some calculations

- Given $a \in [0, L]$, the total pressure exerted on a cross-section

$$S_a^t = \{(a, y, z) \quad \text{s.t.} \quad -A \leq y \leq A, \quad -H(a) \leq z \leq \eta(a, t)\}$$

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- Exercise 1: prove **Archimedes' principle**: the upward buoyant force exerted on a body immersed in a fluid at rest (i.e. $\vec{u} = 0$ and $\eta = 0$) is equal to the weight of the fluid that the body displaces.

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Some calculations

- The mass of water per unit time passing through S_a^t from left to right is given by:

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\begin{aligned}
q_a^t &= \int_{S_a^t} \rho_0 \vec{u}(\gamma, t) \cdot (1,0,0)\, d\gamma \\
&= 2A\rho_0 \int_{-H(a)}^{\eta(a,t)} u_1(a,t)\, dx \\
&= 2A\rho_0 h(a,t) u_1(a,t) \\
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Some calculations

- More generally, if $\varphi(x,t)$ represents the density function of any sustance $\mathcal{S}$ that is transported by the fluid, then the total amount of $\mathcal{S}$ per unit time passing through S_a^t from right to left is given by:

$$
\begin{aligned}
Q_a^{\mathcal{S},t} &= \int_{S_a^t} \varphi(\gamma,t)\vec{u}(\gamma,t)\cdot(1,0,0)\,d\gamma \\
&= 2A\int_{-H(a)}^{\eta(a,t)} \varphi(a,t)u_1(a,t)\,dx \\
&= 2Ah(a,t)\varphi(a,t)u_1(a,t) \\
&= 2Aq_1(a,t)\varphi(a,t).
\end{aligned}
$$

- $Q_a^{\mathcal{S},t}$ is called the **flux** of $\mathcal{S}$.
- The sustance may be a pollutant, the salinity, the internal energy, etc.
- In particular, the flux function of the momentum is as follows:

$$\vec{Q}_x^{m,t} = 2A\rho_0 q_1(x,t)\vec{u}(x,t).$$

Some calculations

- More generally, if $\varphi(x,t)$ represents the density function of any sustance $\mathcal{S}$ that is transported by the fluid, then the total amount of $\mathcal{S}$ per unit time passing through S_a^t from right to left is given by:

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- Mass conservation equation.
  - Given a, b such that $0 \leq a < b \leq L$ and $t > 0$ *the rate of change of the mass of water contained in $\mathcal{O}^t_{a,b}$ is equal to the net flux of mass through the boundaries S_a and S_b.*
  - Using the notation introduced previously, the mathematical expression of this law is as follows:

$$\frac{d}{dt} m^t_{a,b} = q^t_a - q^t_b. \tag{3}$$

  - And using the expressions that have been found:

$$2A\rho_0 \frac{d}{dt} \int_a^b h(x,t)\, dx = 2A\rho_0 q_1(a,t) - 2A\rho_0 q_1(b,t). \tag{4}$$

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- On the other hand, we have:

$$\frac{d}{dt}\int_a^b h(x,t)\,dx = \int_a^b \frac{\partial h}{\partial t}(x,t)\,dx \tag{6}$$

- And using again Barrow's rule:

$$q_1(b,t) - q_1(a,t) = \int_a^b \frac{\partial q_1}{\partial x}(x,t)\,dx. \tag{7}$$

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  - Newton's second law states that mass times acceleration is equal to the total force:
$$m\vec{a} = \vec{F},$$
or, equivalently,
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where $\vec{v}$ is the velocity and $\vec{q}$, the momentum.
  - Coming back to the water flow through the channel, the momentum equation states that *the rate of change of the total momentum contained in $\mathcal{O}^t_{a,b}$ is equal to the net flux of momentum trhough the boundaries S_a and S_b plus the total force*. Using the notation previously introduced:
$$\frac{d}{dt}\vec{M}^t_{a,b} = \vec{Q}^{m,t}_a - \vec{Q}^m_{b,t} + \vec{F},$$
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- These forces are the following:
  - Gravity:
$$\int_a^b \int_{-A}^{A} \int_{-H(x)}^{\eta(x,t)} \rho_0(0,0,-g)\, dz\, dy\, dx = \left(0, 0, -g\, m^t_{a,b}\right).$$
  - The pressure $\vec{P}^t_a$ exerted by the water at the left of $x = a$.
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- Therefore, Newton's second law writes as follows:
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- Using now the expessions of the different terms, we obtain:

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&2A\rho_0 \frac{d}{dt}\left(\int_a^b h(x,t)u_1(x,t)\,dx, 0, \int_a^b \int_{-H(x)}^{\eta(x,t)} u_3(x,z,t)\,dz\,dx\right) = \\
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&+2A\rho_0 g\frac{h(a,t)^2}{2}(1,0,0) - 2A\rho_0 g\frac{h(b,t)^2}{2}(1,0,0) \\
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\frac{d}{dt}\int_a^b q_1(x,t)\,dx &= q_1(a,t)u_1(a,t) - q_1(b,t)u_1(b,t) \qquad (9) \\
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- Taking into account that $q_1 u_1 = q_1^2/h$ it can also be written as follows:

$$\begin{aligned}\frac{d}{dt}\int_a^b q_1(x,t)\,dx &= \frac{q_1^2}{h}(a,t) - \frac{q_1^2}{h}(b,t) \\ &+ \frac{g}{2}h(a,t)^2 - \frac{g}{2}h(b,t)^2 + g\int_a^b h(x,t)H'(x)\,dx.\end{aligned} \tag{10}$$

- By integrating from $t = t_0$ to t_1 and using Barrow's rule, we obtain the **integral form of the momentum equation**:

$$\begin{aligned}\int_a^b q_1(x,t_1)\,dx - \int_a^b q_1(x,t_0)\,dx &= \int_{t_0}^{t_1} \frac{q_1^2}{h}(a,t)\,dt - \int_{t_0}^{t_1} \frac{q_1^2}{h}(b,t)\,dt \\ \int_{t_0}^{t_1} \frac{g}{2}h(a,t)^2\,dt - \int_{t_0}^{t_1} \frac{g}{2}h(b,t)^2\,dt &+ g\int_{t_0}^{t_1}\int_a^b h(x,t)H'(x)\,dx\,dt.\end{aligned} \tag{11}$$

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- If the functions are assumed to be smooth enough, using again Barrow's rule, we obtain from (9):

$$\int_a^b \left(\frac{\partial q_1}{\partial t} + \frac{\partial}{\partial x} \left(\frac{q^2}{h} + \frac{g}{2} h^2 \right) - gh \frac{dH}{dx} \right) dx = 0. \tag{12}$$

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- For simplicity, q_1 and u_1 will be denoted by q and u respectively.

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- For simplicity, q_1 and u_1 will be denoted by q and u respectively.

Conservation Laws

- If the functions are assumed to be smooth enough, using again Barrow's rule, we obtain from (9):

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- Putting together the differential forms of the mass and the momentum equation, we obtain the following PDE system with two equations and two unknowns (h, q):

$$\begin{cases} \dfrac{\partial h}{\partial t} + \dfrac{\partial q}{\partial x} = 0. \\[2ex] \dfrac{\partial q}{\partial t} + \dfrac{\partial}{\partial x}\left(\dfrac{q^2}{h} + \dfrac{g}{2}h^2\right) = gh\dfrac{dH}{dx}. \end{cases} \tag{14}$$

- This system has to be complemented with a set of initial conditions:

$$h(x,0) = h_0(x), \quad q(x,0) = q_0(x), \quad 0 \leq x \leq L,$$

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$$\mathbf{u}_t + \mathbf{f}(\mathbf{u})_x = 0,$$

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$$\mathbf{u} = \begin{pmatrix} h \\ q \end{pmatrix}, \quad \mathbf{f}(\mathbf{u}) = \begin{pmatrix} q \\ q^2/h + gh^2/2 \end{pmatrix}.$$

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- In the general case, it can be written as follows:

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- Exercise 2: Check that the eigenvalues of the Jacobian of the flux function:

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are

$$\lambda_1 = u - \sqrt{gh}, \quad \lambda_2 = u + \sqrt{gh}.$$

- Notice that the eigenvalues are real and different, and thus the PDE system is strictly hyperbolic, if $h > 0$.

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- Nevertheless, this equivalence only holds for smooth solutions. As we shall see, for discontinuous solutions the jump conditions depend on the formulation. And, in this case, the jumps conditions that are consistent with the physics of the problem are those related to the h, q formulation.

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Comments

- The shallow water model can be derived in a rigorous manner from the free surface incompressible Navier-Stokes trhough:
    - A dimensional analysis of the different terms to check those that can be neglected under the long waves assumption.
    - A vertical integration of the simplified 2d system.
- See, for instance: J.F. Gerbeau, B. Perthame, 2001.
- It is not necessary to assume that $u_1 = u_1(x,t)$. The only necessary assumption is that

$$|u_1(x,z,t) - \bar{u}_1(x,t)|$$

is small for every z, where:

$$\bar{u}_1(x,t) = \frac{1}{h(x,t)} \int_{-H(x)}^{\eta(x,t)} u_1(x,z,t)\, dz$$

is the depth-averaged velocity. In fact, the unknowns of the shallow water system are h and $\bar{u}_1$.

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$$\begin{cases} \dfrac{\partial h}{\partial t} + \dfrac{\partial q}{\partial x} = 0. \\[2ex] \dfrac{\partial q}{\partial t} + \dfrac{\partial}{\partial x}\left(\dfrac{q^2}{h} + p(h)\right) = 0. \end{cases} \tag{16}$$

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$$p = \frac{g}{2}h^2. \tag{17}$$

This is the 1d compressible Euler system, where h plays the role of the density and (17) is the law of state.

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$$\frac{\partial u}{\partial x} + \frac{\partial u_3}{\partial z} = 0,$$
and thus:
$$\begin{aligned} u_3(x,z,t) &= u_3(x,-H(x),t) + \int_{-H(x)}^{\eta(x,t)} \frac{\partial u_3}{\partial z}\,dz \\ &= u_3(x,-H(x),t) - \int_{-H(x)}^{z} \frac{\partial u}{\partial x}\,dz \\ &= u_3(x,-H(x),t) - (z+H(x))\frac{\partial u}{\partial x}. \end{aligned}$$
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Dimensionless form

- Let us consider some characteristic values x^*, z^*, t^*, u^*, of the variables x, z, t, u. For instance x^* is the length of the channel L; z^*, the mean depth of the channel; u^*, the characteristic horizontal speed of the water; and

$$t^* = \frac{x^*}{u^*}.$$

- We consider the change of variables:

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Some easy computations allow us to obtain the expression of the system in the dimensionaless variables:

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$$Fr = \frac{\sqrt{gz^*}}{u^*}$$

  is the so-called **Froude number**.

- The basic hypothesis to derive the shallow water system from the incompressible Euler or Navier-Stokes system is

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- The general expression of the hyperbolic 2d shallow water model is as follows:

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Friction forces

- The expression of the friction forces is given frequently by an empiric form. Usually it takes the form of a quadratic law:
$$-C(h)|\vec{u}|\vec{u}.$$
- Some possible expressions are the following:
  - Chézy law: $-\frac{g}{hC^2}\cdot|\vec{u}|\vec{u}.$
  - Strickler law: [illegible]
  - Manning law: [illegible]
- The friction with the air can also be taken into account using a quadratic expression of the wind velocity above the boundary layer.
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Friction forces

- The expression of the friction forces is given frequently by an empiric form. Usually it takes the form of a quadratic law:
$$-C(h)|\vec{u}|\vec{u}.$$
- Some possible expressions are the following:
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Scalar conservation laws

- Let us consider a Cauchy problem of the form:

$$\begin{cases} u_t + f(u)_x = 0, & x \in \mathbb{R}, t \geq 0; \\ u(x,0) = u_0(x), & x \in \mathbb{R}; \end{cases}$$

  where:
  - $f : \mathbb{R} \to \mathbb{R}$ is a smooth function;
  - $u_0 : \mathbb{R} \to \mathbb{R}$ is a known function.
- If u is a solution and the equation is integrated in a rectangle $[a,b] \times [t_0, t_1]$ we obtain the **integral form** of the equation;

$$\int_a^b u(x,t_1)dx = \int_a^b u(x,t_0)dx + \int_{t_0}^{t_1} f(u(a,t))dt - \int_{t_0}^{t_1} f(u(b,t))dt.$$

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$$\int_{\mathbb{R}} u(x,t_1)dx = \int_{\mathbb{R}} u(x,t_0)dx.$$

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Scalar conservation laws: examples

- **Burgers equation:**

$$u_t + \left(\frac{u^2}{2}\right)_x = 0.$$

- **Traffic models:**

$$u_t + (v \cdot u)_x = 0.$$

  - $u(x,t)$ traffic density;
  - $v = v(u)$ traffic velocity.
  - Example:

$$v(u) = \frac{v_{max}}{\beta}(\beta - u),$$

  where v_{max} is the maximal speed at the road and β is the saturation density.

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Example: linear scalar conservation law

- Let us consider the particular case $f(u) = au$, $a \in \mathbb{R}$:

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- If u is a solution, then u remains constant along every straight line of the family $x = at + k, k \in \mathbb{R}$. In effect:

$$\frac{d}{dt}(u(at+k,t)) = a\frac{\partial u}{\partial x}(at+k,t) + \frac{\partial u}{\partial t}(at+k,t) = 0.$$

- These straight lines are the so-called **characteristic curves** of the PDE. Given an initial condition u_0, they allow us to easily compute the corresponding solution:

$$u(x,t) = u_0(x - at), \quad \forall x \in \mathbb{R}, t \geq 0.$$

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Characteristic curves

- Let us come back to the general case:

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- The conservation law may be rewritten as follows:

$$u_t + a(u)u_x = 0, \quad x \in \mathbb{R}, t > 0,$$

where $a(u) = f'(u)$.

- $a(u)$ is expected to play a similar role to a in the scalar case. This fact suggests to define the characteristic curves as the family of integral curves of the ODE:

$$\frac{dx}{dt} = a(u(x,t)).$$

- Notice that, unlike the linear case, the characteristic cruves depend on the particular solution considered, as u explicitly appears in the expression of the ODE.

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$$\frac{d}{dt}(u(x(t),t)) = \left(\frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial t}\right)(x(t),t) = \left(\frac{\partial u}{\partial x}a(u) + \frac{\partial u}{\partial t}\right)(x(t),t) = 0.$$

- Given a solution u, the characteristic passing by x_0 at time $t = 0$ is the solution of the Cauchy problem:

$$\begin{cases} \dfrac{dx}{dt} = a(u(x,t)); \\ x(0) = x_0. \end{cases} \tag{20}$$

- But we know that u is constant through the curve, i.e.

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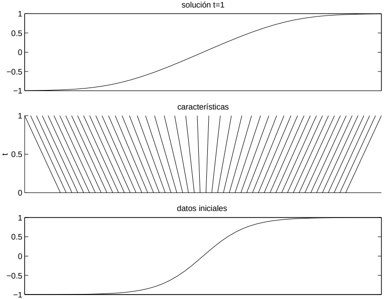
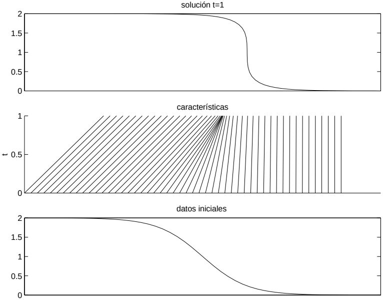


- $u(x,0) = 1 - \tanh(x)$.



- The characteristic curves are divergent and the solution is smooth $\forall t$.
- The characteristic are divergent and the solution becomes singular in a finite time.

Weak solutions

Scalar conservation laws

- Goal: to introduce a definition of weak solution allowing the solutions to become discontinuous.
- A function $u : \mathbb{R} \times \mathbb{R}^+ \to \mathbb{R}$ is said to be piecewise $\mathcal{C}^1$ if there exists a finite number of smooth curves $\Gamma_1,\dots,\Gamma_N$ outside which u is $\mathcal{C}^1$ and across which u has a jump discontinuity. Moreover, we assume that any line of discontinuity Γ_i has a a parameterization of the form:

$$\Gamma_i = \{(\sigma_i(t), t), t \in I_i\},$$

where I_i is an interval of $[0, \infty)$ and $\sigma_i : I_i \subset \mathbb{R} \to \mathbb{R}$ a $C^1(I_i)$ function. Let us denote by $u^\pm$ the lateral limits of u at any point of a curve Γ_i:

$$\lim_{x \to \gamma^+} u(x, t_0) = u^+(\gamma, t_0), \qquad \lim_{x \to \gamma^-} u(x, t_0) = u^-(\gamma, t_0).$$

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$$\forall (a,b) \subset \mathbb{R},\ \forall (t_0,t_1) \subset \mathbb{R}^+.$$

- If u is a piecewise C^1 function, the integrals in $[a,b]$ appearing in the definition make sense.
- Nevertheless, there is a difficulty where the time integrals whenever there are a stationary discontinuity at $x = a$ or $x = b$. This difficulty will be discussed later on.

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Weak solutions: example

- Example: let us look for weak solutions of the form:

$$u(x,t) = \begin{cases} u_L & \text{if } x < \sigma(t); \\ u_R & \text{if } x > \sigma(t); \end{cases}$$

where u_L y u_R are two real numbers and $\sigma : [0, \infty) \to \mathbb{R}$ a differentiable function.

- The equality

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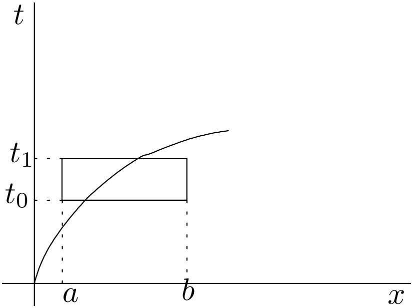

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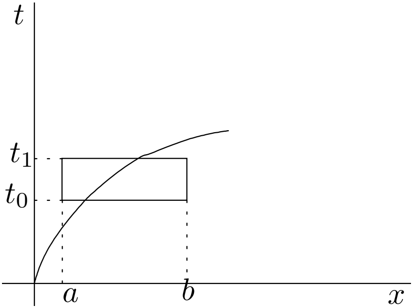

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- Let us assume that u is a smooth solution and that φ is a $\mathcal{C}^1$ function with a compact support, i.e. $\varphi \in \mathcal{C}^1_0\left(\mathbb{R} \times [0, \infty)\right)$.
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$$\begin{aligned} 0 &= \int_{\mathbb{R}\times\mathbb{R}^+} \left(\frac{\partial u}{\partial t} + \frac{\partial}{\partial x} f(u) \right) \varphi \, dx \, dt \\ &= \int_{\mathbb{R}} \left(\int_{\mathbb{R}^+} \frac{\partial u}{\partial t} \varphi \, dt \right) dx + \int_{\mathbb{R}^+} \left(\int_{\mathbb{R}} \frac{\partial}{\partial x} f(u) \varphi \, dx \right) dt \\ &= -\int_{\mathbb{R}\times\mathbb{R}^+} \left(u \frac{\partial \varphi}{\partial t} + f(u) \frac{\partial \varphi}{\partial x} \right) dx \, dt - \int_{\mathbb{R}} u(x,0)\varphi(x,0) \, dx. \end{aligned}$$

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- If u has a stationary contact discontinuity at $x = a$, the Rankine Hugoniot condition reduces to:

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  Therefore, $f(u)$ is continuous and thus the time integral

  $$\int_{t_0}^{t_1} f(u(a,t))\, dt$$

  appearing at Definition 1 is not ambiguous.

- It is possible to construct two conservation laws whose smooth solutions are the same but whose weak solutions are different. In other words, if a conservation law is reformulated using mathematical operations that are only valid for smooth solutions, it may happen that the corresponding Rankine-Hugoniot condition, and thus the weak solutions, are not equivalent.

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Sel-similar solutions

- A function u is said to be a self-similar solution of the conservation law if it can be written in the form
$$u(x,t) = v\left(\frac{x - x_0}{t - t_0}\right)$$
for some $x_0 \in \mathbb{R}$, $t_0 \leq 0$ and a continuous function $v : \mathbb{R} \to \mathbb{R}$.
- If u is a self-similar solution, it is constant along the straight lines of the family:
$$\frac{x - x_0}{t - t_0} = C.$$
- For simplicity, let us take $x_0 = t_0 = 0$:
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$$v'\left(\frac{x}{t}\right)\left[a\left(v\left(\frac{x}{t}\right)\right) - \frac{x}{t}\right] = 0 \quad \forall x \in \mathbb{R}, \quad \forall t > 0.$$

- The function v has to satisfy:

$$v'(\xi)\left(a(v(\xi)) - \xi\right) = 0, \quad \forall \xi \in \mathbb{R}.$$

- Either $v'(\xi) = 0$ or $a(v(\xi)) = \xi$, i.e. either v is constant or it is the inverse function of a.
- Given u_L, u_R such that $a(u_L) < a(u_R)$, if a has an inverse function in $[a(u_L), a(u_R)]$, then the function:

$$u(x,t) = \begin{cases} u_L & \text{if} \quad x < a(u_L)t; \\ a^{-1}(x/t) & \text{if} \quad a(u_L)t < x < a(u_R)t; \\ u_R & \text{if} \quad x > a(u_R)t, \end{cases}$$

is a self-similar solution.

Self-similar solutions

- Let us look for continuous self-similar solutions $u(x,t) = v(x/t)$:

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Self-similar solutions

- If, in particular, f is strictly convex or concave, $a(u) = f'(u)$ is strictly monotone and thus it has an inverse function at every interval. Therefore, a self-similar solution linking u_L and u_R can be constructed whenever $a(u_L) < a(u_R)$, i.e.
  - when $u_L < u_R$, if f is strictly convex;
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- These self-similar solutions are called rarefaction waves.
- In the case of Burgers equation, $a(u) = u$. Given two states u_L and u_R such that $u_L < u_R$, they can be linked by the rarefaction wave:

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Non-uniqueness of weak solution

- Let us consider Burgers equation with the initial condition:

$$u_0(x) = \begin{cases} -1 & \text{if} \quad x < 0; \\ 1 & \text{if} \quad x > 0; \end{cases}$$

- A solution of this Cauchy problem is the rarefaction wave.
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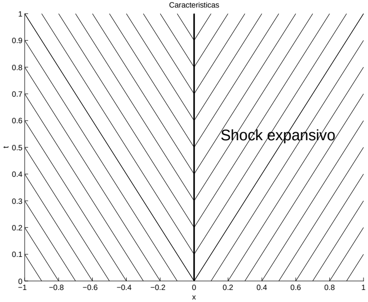
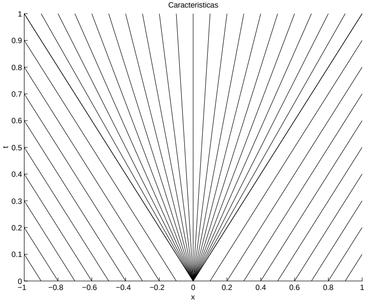
$$u(x,t) = \operatorname{sign}(x) = \begin{cases} -1 & x < 0 \\ 1 & x \geq 0 \end{cases} \qquad v(x,t) = \begin{cases} -1 & x/t \leq -1 \\ x/t & -1 < x/t < 1 \\ 1 & x/t \geq 1 \end{cases}$$



Stationary discontinuity



Rarefaction wave

Entropy condition

- It is necessary to have a criterion to decide whether a weak solution is physically meaningful or not.
- Lax's entropy condition: A discontinuity of a piecewise C^1 solution u is said to be **admissible** if the Rankine-Hugoniot conditions are satisfied and

$$a(u^-) \geq s \geq a(u^+)$$

where u^- and u^+ are the left and right limits at the discontinuity and s, the speed of propagation of the discontinuity.
- Definition 1: A weak solution is said to be an **entropy solution** if it is continuous or if its discontinuities are admissible.
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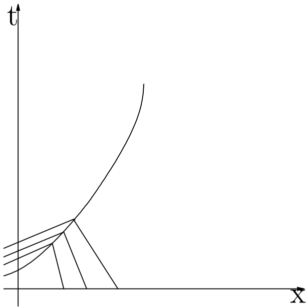
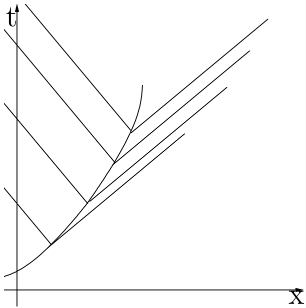


Admissible discontinuity



Non-admissible discontinuity

Entropy conditions

- Vanishing viscosity method:
  - Let us consider a parabolic regularization of the conservation law:

$$\frac{\partial u_\varepsilon}{\partial t} + \frac{\partial}{\partial x} f(u_\varepsilon) = \varepsilon \frac{\partial^2 u_\varepsilon}{\partial x^2}.$$

  - The entropy solutions can be defined as those that can be obtained as a limit of a sequence u_ε of solutions of the parabolic regularization as ε goes to 0.
  - Definition: a pair of functions (η, G) from $\mathbb{R}$ to $\mathbb{R}$ is said to be an **entropy pair** if:
    - η, the so-called entropy function, is strictly convex;
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- It can be shown that the solutions of the parabolic regularizations satisfy the inequality

$$\eta(u_\varepsilon)_t + G(u_\varepsilon)_x \leq \varepsilon \frac{\partial^2}{\partial x^2}\eta(u_\varepsilon).$$

- Definition 2: A weak solution is said to be an **entropy solution** if it satisfies the inequality:

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  in the distributions sense.

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The Riemann Problem

- Given a scalar conservation law, a Riemann Problem is a Cauchy problem whose initial condition has the form:

$$u(x,0) = \begin{cases} u_L & x < 0 \\ u_R & x \geq 0 \end{cases}$$

- If f is strictly convex of concave, the solution of the Riemann problem is given by:
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$$u(x,t) = \begin{cases} u_L & \text{if} \quad x < a(u_L)t; \\ a^{-1}(x/t) & \text{if} \quad a(u_L)t < x < a(u_R)t; \\ u_R & \text{if} \quad x > a(u_R)t, \end{cases}$$

  - If $u_L = u_R$ the solution is constant.
  - If $a(u_L) > a(u_R)$ the solution is shock wave travelling at speed:

$$s = f(u_R) - f(u_L)/u_R - u_L$$

  i.e.

$$u(x,t) = \begin{cases} u_L & \text{if} \quad x < s\,t; \\ u_R & \text{if} \quad x > s\,t. \end{cases}$$

The Riemann Problem

- Notice that the solution of the Riemann problem is self-similar in all the cases. In what follows it will denoted by:

$$u(x,t) = V\left(\frac{x}{t}; u_L, u_R\right).$$

- As a consequence, observe that the solution of the Riemann problem at $x = 0$ is constant.

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The linear case

- Let us consider the PDE system:

$$\mathbf{u}_t + \mathcal{A}\mathbf{u}_x = 0,$$

where

$$\mathbf{u}(x,t) = \begin{bmatrix} u_1(x,t) \\ \vdots \\ u_N(x,t) \end{bmatrix},$$

and $\mathcal{A}$ is a constant $N \times N$ matrix.

- Let us assume that the system is strictly hyperbolic, i.e. $\mathcal{A}$ has N different real eigenvalues $\lambda_1 < \cdots < \lambda_N$. Let $\{\mathbf{r}_1, \ldots, \mathbf{r}_N\}$ be a basis of associated eigenvectors.

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The linear case

- Let $\mathcal{K}$ be the matrix whose columns are given by the eigenvectors $\mathbf{r}_i$. One has:

$$\mathcal{A} = \mathcal{K}\Lambda\mathcal{K}^{-1},$$

  where Λ is the diagonal matrix whose coefficients are the eigenvalues.

- If the change of variables

$$\mathbf{v} = \mathcal{K} \cdot \mathbf{u},$$

  is applied, the system reduces to N uncoupled scalar linear conservation laws:

$$\frac{\partial v_i}{\partial t} + \lambda_i \frac{\partial v_i}{\partial x} = 0, i = 1, \ldots, N.$$

- The components of $\mathbf{v}$ are called the characteristic variables, while those of of $\mathbf{u}$ are called the conserved variables.

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- To solve the Cauchy problem with initial condition

$$\mathbf{u}(x,0) = \mathbf{u}_0(x),$$

1. First, the expression of the initial condition in the eigenvector basis is computed:

$$\mathbf{u}_0(x) = \sum_{i=1}^{N} v_i^0(x)\mathbf{r}_i.$$

2. Next, the uncoupled conservation laws are solved:

$$v_i(x,t) = v_i^0(x - \lambda_i t), \quad i = 1, \ldots, N.$$

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- In particular, the solution of the Riemann problem whose initial conditions are given by two states $\mathbf{u}_L$, $\mathbf{u}_R$ consist of N discontinuities issuing from the origin and travelling at constant speeds $\lambda_1, \ldots, \lambda_N$.
- These discontinuities link $N+1$ states:

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- The Jacobian matrix of the flux function F is given by:

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Eigenvalues, characteristic fields

- Let us assume that the system is strictly hyperbolic: for every state U, $\mathcal{A}(\mathbf{u})$ has N different real eigenvalues $\lambda_1(\mathbf{u}) < \cdots < \lambda_N(\mathbf{u})$. Let $\mathbf{r}_1(\mathbf{u}), \ldots, \mathbf{r}_N(\mathbf{u})$ be associated eigenvectors.
- $\mathbf{r}_1(\mathbf{u}), \ldots, \mathbf{r}_N(\mathbf{u})$ are called the **characteristic fields**.
- If

$$\nabla \lambda_i(\mathbf{u}) \cdot \mathbf{r}_i(\mathbf{u}) \neq 0, \quad \forall \mathbf{u},$$

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Speed of propagation of small waves

- The eigenvalues are the speed of propagation of small waves: let us consider a sinusoidal perturbation of a constant state $\mathbf{u}_0$:

$$\mathbf{u}_0 + \bar{\mathbf{u}}\cos(\omega t),$$

which is small in the sense that $\bar{\mathbf{u}} << \mathbf{u}_0$.

- Let us look for a solution of the form:

$$\mathbf{u}(x,t) = \mathbf{u}_0 + \bar{\mathbf{u}}\cos(\omega(x-ct)),$$

i.e. the perturbation propagating at speed c.

- One has:

$$\begin{aligned} 0 &= \mathbf{u}_t + \mathbf{f}(\mathbf{u})_x \\ &= \mathbf{u}_t + \mathcal{A}(\mathbf{u})\mathbf{u}_x \\ &= (-c\bar{\mathbf{u}} + \mathcal{A}(\mathbf{u})\bar{\mathbf{u}})\,\omega\sin(\omega(x-ct)) \\ &\cong (-c\bar{\mathbf{u}} + \mathcal{A}(\mathbf{u}_0)\bar{\mathbf{u}})\,\omega\sin(\omega(x-ct)). \end{aligned}$$

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Speed of propagation of small waves, jump condition

- We find an approximate solution if there exists $i \in \{1, \dots, N\}$ such that

$$c = \lambda_i(\mathbf{u}_0)$$

  and $\bar{\mathbf{u}}$ is an associated eigenvector.

- The Rankine-Hugoniot condition writes as follows:

$$[\mathbf{f}(\mathbf{u})] = s[\mathbf{u}],$$

  where s is the speed of propagation of the shock.

- Lax's entropy condition: a shock is admissible if there exists $i \in \{1, \dots, N\}$ such that:

$$\lambda_i(\mathbf{u}^-) > s > \lambda_{i-1}(\mathbf{u}^-), \quad \lambda_{i+1}(\mathbf{u}^+) > s > \lambda_i(\mathbf{u}^+)$$

  if the i-th characteristic field is genuinely nonlinear, or

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Characteristic curves, simple waves, Riemann problem

- Given a solution $\mathbf{u}$, the characteristic curves are the integral curves of the N ODE systems:

$$\frac{dx}{dt} = \lambda_i\big(\mathbf{u}(x(t),t)\big), \quad i = 1, \dots, N.$$

- Associated to every family of characteristic curves, and depending of their nature, one can construct solutions that are:
  - contact discontinuities that propagate following a characteristic curve, as in the linear scalar case, if the characteristic field is linearly degenerate;
  - rarefaction waves and entropy shocks, if the characteristic field is genuinely nonlinear.
- These solutions are called simple waves.
- If all the characteristic fields are either genuinely nonlinear or linearly degenerate, the solution of the Riemann problem consists of a N simple waves linking $N+1$ intermediate states.

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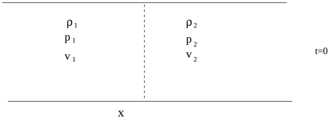
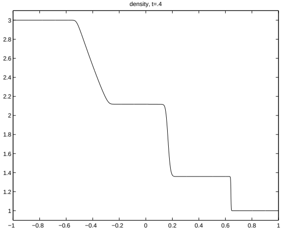
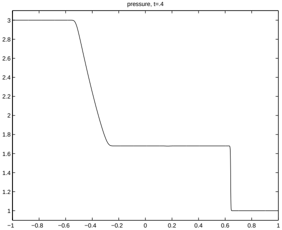
Example

Example: Euler system with initial conditions

$$\rho(x,0)=\begin{cases}\rho_1 & x<0\\ \rho_2 & x>0\end{cases}\qquad v(x,0)=\begin{cases}v_1 & x<0\\ v_2 & x>0\end{cases}\qquad p(x,0)=\begin{cases}p_1 & x<0\\ p_2 & x>0\end{cases}$$







The shallow water model

- In the particular case of the shallow water system:

$$\mathbf{u} = \begin{pmatrix} h \\ q \end{pmatrix}, \quad \mathbf{f}(\mathbf{U}) = \begin{pmatrix} q \\ q^2/h + gh^2/2 \end{pmatrix}.$$

$$\frac{\partial \mathbf{f}}{\partial \mathbf{u}} = \begin{pmatrix} 0 & 1 \\ -\frac{q^2}{h^2} + gh & 2\frac{q}{h} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -u^2 + gh & 2u \end{pmatrix}$$

$$\lambda_1(\mathbf{u}) = u - \sqrt{gh}, \quad \lambda_2(\mathbf{u}) = u + \sqrt{gh}.$$

- The eigenvalues are real and different if $h > 0$. They are the speed of propagation of small waves.
- Exercise 4: Compute the characteristic fields and check that they are genuinely non-linear.
- The solutions of the Riemann problems consist of two simple waves (rarefactions and/or shock waves).

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$$\mathbf{u} = \begin{pmatrix} h \\ q \end{pmatrix}, \quad \mathbf{f}(\mathrm{U}) = \begin{pmatrix} q \\ q^2/h + gh^2/2 \end{pmatrix}.$$

$$\frac{\partial \mathbf{f}}{\partial \mathbf{u}} = \begin{pmatrix} 0 & 1 \\ -\frac{q^2}{h^2} + gh & 2\frac{q}{h} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -u^2 + gh & 2u \end{pmatrix}$$

$$\lambda_1(\mathbf{u}) = u - \sqrt{gh}, \quad \lambda_2(\mathbf{u}) = u + \sqrt{gh}.$$

- The eigenvalues are real and different if $h > 0$. They are the speed of propagation of small waves.
- Exercise 4: Compute the characteristic fields and check that they are genuinely non-linear.
- The solutions of the Riemann problems consist of two simple waves (rarefactions and/or shock waves).

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The shallow water model

- Let Fr be the Froude number:
$$Fr = \frac{|u|}{\sqrt{gh}}.$$
- If $Fr > 1$, then both eigenvalues $u \pm \sqrt{gh}$ have the same sign of u: the small waves travel in the flow sense. The flow is said to be **supercritical** (is the equivalent concept to supersonic in the case of a compressible gas.
- If $Fr < 1$ the signs of the eigenvalues are different: some smalls waves traves in the flow sense and some other in the other sense. The flow is said to be **subcritical**.
- If $Fr = 1$ the flow is said to be **critical**.

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- The Rankine-Hugoniot coniditions are:

$$\begin{aligned} s[h] &= [q], \\ s[q] &= \left[\frac{q^2}{h} + g\frac{h^2}{2}\right] \end{aligned}$$

- The first equation can be rewritten as follows:

$$v^+h^+ = v^-h^-,$$

where

$$v^+ = u^+ - s, \quad v^- = u^- - s$$

are the speed of the particles relative to the shock speed at both sides of the shock. Therefore, both relative speeds have the same sign.
- The fluid particle cross the shock from left to right if $v^{\pm} > 0$ and from right to left if $v^{\pm} < 0$.
- It can be shown that the entopy condition is equivalent to:

$$\begin{aligned} h^- &< h^+ \quad \text{if } v^{\pm} > 0, \\ h^+ &< h^- \quad \text{if } v^{\pm} < 0, \end{aligned}$$

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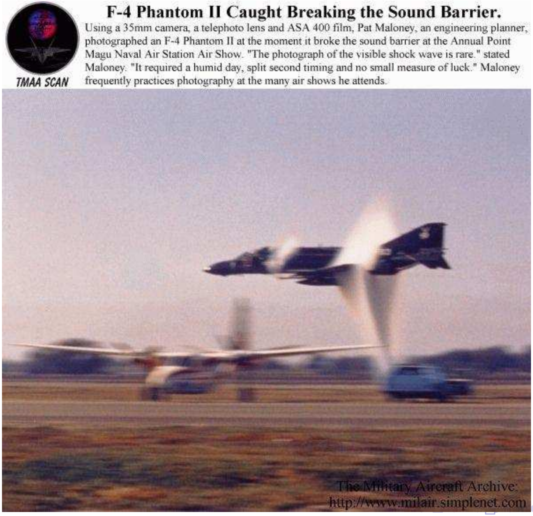

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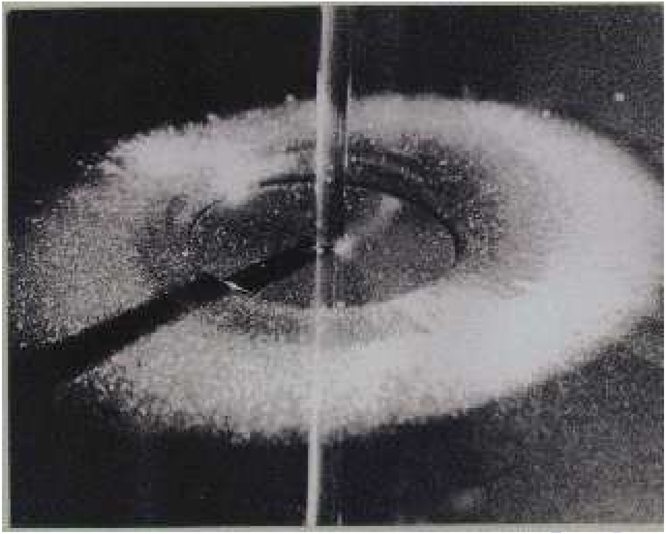

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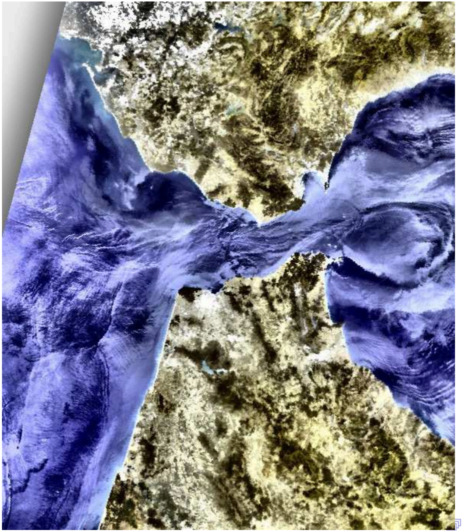


Shocks in shallow waters

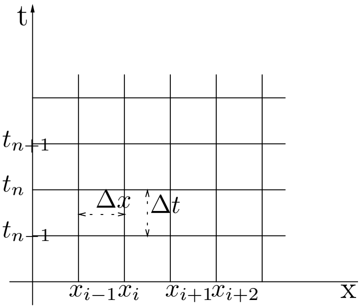
- Internal tides in the Strait of Gibraltar.

Finite differences

- Let us consider the Cauchy problem:

$$\begin{cases} \dfrac{\partial u}{\partial t} + a\,\dfrac{\partial u}{\partial x} = 0, \quad x \in \mathbb{R}, t > 0; \\ \\ u(x,0) = u_0(x), \quad x \in \mathbb{R}. \end{cases}$$

- For simplicity, we consider a uniform mesh of the half-plane $t \geq 0$.



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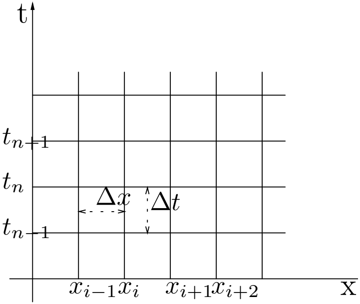

Finte differences

- The partial derivatives appearing at the equation are approached using only points of the mesh.
- Space partial derivatives:

$$\begin{aligned}
\frac{\partial u}{\partial x}(x_i,t_n) &= \frac{u(x_{i+1},t_n)-u(x_i,t_n)}{\Delta x}+O(\Delta x),\\
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- A smooth exact solution has to satisfy:

$$\frac{u(x_i, t_{n+1}) - u(x_i, t_n)}{\Delta t} + a\,\frac{u(x_{i+1}, t_n) - u(x_i, t_n)}{\Delta x} \cong 0.$$

- The idea of the method is to look for approximations that exactly satisfy this equation u_i^n de $u(x_i, t_n)$:

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$$u_i^{n+1} = u_i^n - a\,\frac{\Delta t}{\Delta x}(u_{i+1}^n - u_i^n).$$

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- The two other discretization of the space derivative lead to the methods:
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Las's equivalence theorem

- Theorem: For linear problems, a consistent numerical method is convergent if and only if it is stable.
- Consistency: The exact values of a smooth solutions at the mesh points have to satisfy the finite difference equation with errors that converge to 0 as Δx, Δt tend to 0. These errors are the so-called local discretization errors.
  - For instance, for the forward difference method:

$$L_i^n = \frac{u(x_i, t_{n+1}) - u(x_i, t_n)}{\Delta t} + a\frac{u(x_{i+1}, t_n) - u(x_i, t_n)}{\Delta x} = O(\Delta x + \Delta t).$$

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  - A method is said to be of order (p, q) if:
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  - The orders of the backward, forward, and centered methods are, respectively, (1,1), (1,1), and (2,1)

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Stability

- Stability: The local discretization errors and the round-off errors are not dramatically amplified as the numerical method advances in time.
  - As the exact solution satisfies:
  $$\sup\{|u(x,t)|,\ x \in \mathbb{R}\} = \sup\{|u_0(x)|,\ x \in \mathbb{R}\}, \quad \forall t > 0,$$
  a reasonable stability criterium is the following: given $T > 0$, there exists $K > 0$ independent of Δt and Δx such that:
  $$\sup_{i \in \mathbb{Z}} |u_i^n| \leq K \sup_{i \in \mathbb{Z}} \left| u_i^0 \right|, \quad \forall n \leq \frac{T}{\Delta t}.$$
  - Let u^n be the piecewise constant function taking the value u_i^n in $[x_i - \Delta x/2, x_i + \Delta x/2)$. Using this function, the previous inequality writes as follows:
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  - It is possible to give an alternative definition of stability taking any other functional norm, for instance the L^p norm $||\cdot||_p$.
  - In particular, the L^2 norm $||\cdot||_2$ allows use to use Von Neumann stability analysis based on the Fourier transform. This analysis is very convenient and useful for linear problems, but its generalization to nonlinear ones is not easy.
- Example: Stability of the backward difference method:

$$\begin{aligned}\left|u_i^{n+1}\right| &= \left|u_i^n(1-a\frac{\Delta t}{\Delta x}) + a\frac{\Delta t}{\Delta x}u_{i-1}^n\right| \\ &\leq |u_i^n|\left|1-a\frac{\Delta t}{\Delta x}\right| + |a|\frac{\Delta t}{\Delta x}|u_{i-1}^n| \\ &\leq \left(\left|1-a\frac{\Delta t}{\Delta x}\right| + |a|\frac{\Delta t}{\Delta x}\right)\left(\sup_{j\in\mathbb{Z}}|u_j^n|\right).\end{aligned}$$

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$$\begin{aligned} \left|u_i^{n+1}\right| &= \left|u_i^n(1 - a\frac{\Delta t}{\Delta x}) + a\frac{\Delta t}{\Delta x}u_{i-1}^n\right| \\ &\leq \left|u_i^n\right|\left|1 - a\frac{\Delta t}{\Delta x}\right| + |a|\frac{\Delta t}{\Delta x}\left|u_{i-1}^n\right| \\ &\leq \left(\left|1 - a\frac{\Delta t}{\Delta x}\right| + |a|\frac{\Delta t}{\Delta x}\right)\left(\sup_{j\in\mathbb{Z}}\left|u_j^n\right|\right). \end{aligned}$$

Stability

- Comments:
  - It is possible to give an alternative definition of stability taking any other functional norm, for instance the L^p norm $||\cdot||_p$.
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- If $a > 0$ and the CFL condition (Courant, Friedrichs, Levy):

$$|a|\,\frac{\Delta t}{\Delta x} \leq 1,$$

is satisfied, we obtain

$$|u_i^{n+1}| \leq \sup_{j\in\mathbb{Z}} |u_j^n|$$

and thus:

$$\sup_{j\in\mathbb{Z}} |u_j^n| \leq \sup_{j\in\mathbb{Z}} |u_j^0|, \quad \forall n.$$

The backward difference method is condicionally stable for $a > 0$. It is unstable if $a < 0$.

- Analogously, the forward difference method is stable if $a < 0$ and the CFL condition is satisfied. It is unstable if $a > 0$.
- Finally, the centered method is unconditionally unstable.

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Stability

- Notice that stability is achieved by **upwinding** and by restricting the time step, so that the numerical dependency domain contains the analytical one.
- For instance, in the case of the forward difference method, the value of u_i^n only depends on the initial data in the nodes belonging to the interval

$$[x_i, x_{i+n}].$$

- On the other hand, the value of the exact solution only depends on the initial data in $x_i - at_n$.
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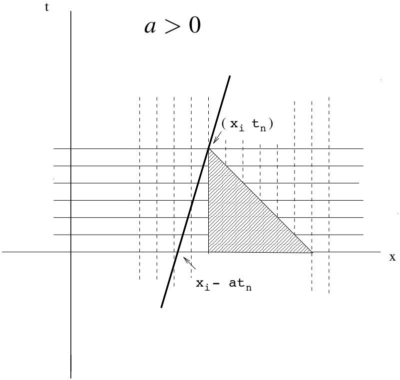

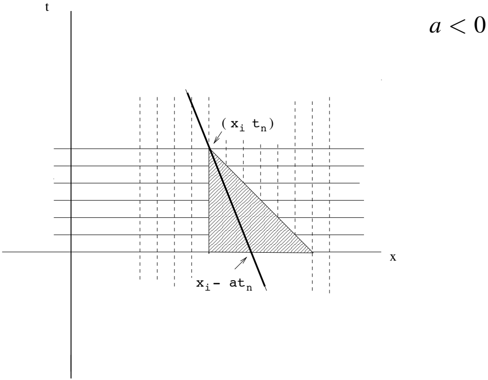

The upwind method: viscous form

- Using the notation
$$a^+ = max\{a, 0\}, \quad a^- = min\{a, 0\}$$
the backward and forward method may be combined in a unique expression:
$$u_i^{n+1} = u_i^n - \frac{\Delta t}{\Delta x}\{a^+(u_i^n - u_{i-1}^n) + a^-(u_{i+1}^n - u_i^n)\},$$
which is the so-called CIR (after Courant, Isaacson y Rees) or Upwind method.
- Using the equalities:
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$$u_i^{n+1} = u_i^n - a\,\frac{\Delta t}{2\Delta x}\left(u_{i+1}^n - u_{i-1}^n\right) + |a|\,\frac{\Delta t}{2\Delta x}\left(u_{i+1}^n - 2u_i^n + u_{i-1}^n\right).$$

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- A general method in viscous form is as follows:

$$u_i^{n+1} = u_i^n - a\,\frac{\Delta t}{2\Delta x}\left(u_{i+1}^n - u_{i-1}^n\right) + \frac{q}{2}\left(u_{i+1}^n - 2u_i^n + u_{i-1}^n\right).$$

- It can be interpreted as the centered method plus the discretization of a second order space derivative:

$$\frac{q}{2}\frac{\Delta x}{\Delta t}\Delta x\frac{\partial^2 u}{\partial x^2}.$$

This is the so-called **numerical viscosity**.

- The numerical viscosity stabilizes the centered scheme if the parameter q is adequately chosen: it can be shown that the numerical method is L^2-stable if and only if q is such that:

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Lax-Wendroff and Lax-Friedrichs method

- The minimal choice of q leads to the **Lax-Wendroff method**:

$$u_i^{n+1} = u_i^n - a\,\frac{\Delta t}{2\Delta x}\left(u_{i+1}^n - u_{i-1}^n\right) + \frac{1}{2}\left(\frac{a\,\Delta t}{\Delta x}\right)^2\left(u_{i+1}^n - 2u_i^n + u_{i-1}^n\right).$$

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- It can be rewritten as follows:

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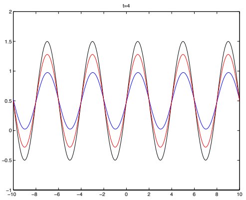
Comparison of numerical methods for smooth solutions

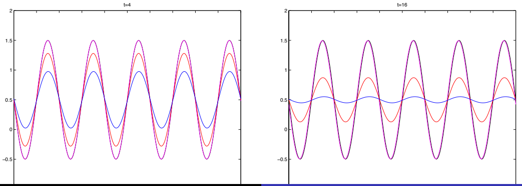
First order methods: u_0, u^n-LxF, u^n-UpW.

$t = 4$ $t = 8$



Second order methods:LxW.

$t = 4$ $t = 16$



Comparison of numerical methods for discontinuous solutions

- A convergente numerical method can produce wrong numerical approximation of discontinuous solutions.

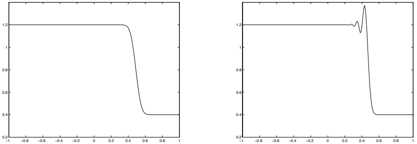


Figure: Numerical solution obtained with the upwind method (left) and the Lax-Wendroff method (right) for the linear equation with $a = 1$, $\Delta x = 0.01$, $\Delta t = 0.005$

Nonlinear problems

- Let us consider again the general problem:

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x} f(u) = 0, \quad x \in \mathbb{R}, t \geq 0;$$

- If the equation is written in the form:

$$\frac{\partial u}{\partial t} + a(u)\frac{\partial u}{\partial x} = 0,$$

  where $a(u) = f'(u)$, it is clear that $a(u)$ locally plays the same role of a for the linear problem.
- In particular, the following extension of the upwind method is natural:

$$u_i^{n+1} = u_i^n - \frac{\Delta t}{\Delta x}\{a(u_i^n)^+(u_i^n - u_{i-1}^n) + a(u_i^n)^-(u_{i+1}^n - u_i^n)\}.$$

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Finite volume methods

- We define the cells or finite volumes $\{I_i\}$ como

$$I_i = [x_{i-1/2}, x_{i+1/2}]$$

where $x_{i-1/2} = x_i - \Delta x/2$ y $x_{i+1/2} = x_i + \Delta x/2$.

- Weak solutions satisfy:

$$\frac{1}{\Delta x}\int_{I_i} u(x,t_{n+1})\,dx = \frac{1}{\Delta x}\int_{I_i} u(x,t_n)\,dx$$
$$+\frac{\Delta t}{\Delta x}\left\{\frac{1}{\Delta t}\int_{t_n}^{t_{n+1}} f(u(x_{i-1/2},t))\,dt - \frac{1}{\Delta t}\int_{t_n}^{t_{n+1}} f(u(x_{i+1/2},t))\,dt\right\}.$$

- u_i^n will be interpreted as an approximation of the average of u at the cell I_i at the instant t_n:

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- It is natural to consider numerical methods of the form:

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where:

$$F_{i+1/2}^n = F(u_{i-q}^n, u_{i-q+1}^n, \dots, u_{i+p-1}^n, u_{i+p}^n).$$

is a consistent approximation of the averaged flow through the intercell $x_{i+1/2}$ between the times t_n and t_{n+1}:

$$F_{i+1/2}^n \cong \frac{1}{\Delta t}\int_{t_n}^{t_{n+1}} f(u(x_{i+1/2},t))dt.$$

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- If the numerical flux F is a Lispchitz-continuous and satisfies:

$$F(u,u,\ldots,u)=f(u),$$

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- The Lax-Wendroff's Theorem ensures that, if the numerical approximation provided by a consistent conservative method converge, the limit is a weak solution of the equation.

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- In the case of a linear equation, all the finite difference methods previously introduced can be interpreted as conservative metidos. In particular the centered, upwind, Lax-Wendroff, and Lax-Friedrichs methods are equivalent respectively to the conservative methods corresponding to the numerical fluxes:

$$\begin{aligned}
F^{cen}(u,v) &= a\,\frac{u+v}{2}, \\
F^{cir}(u,v) &= a\,\frac{u+v}{2} - \frac{|a|}{2}(u-v), \\
F^{LW}(u,v) &= a\,\frac{u+v}{2} - \frac{a^2\Delta t}{2\Delta x}(u-v), \\
F^{LF}(u,v) &= a\,\frac{u+v}{2} - \frac{\Delta x}{2\Delta t}(u-v).
\end{aligned}$$

Finite volume methods

- It is easy to generalize these numerical fluxes to the general case:

$$
\begin{aligned}
F^{cen}(u,v) &= \frac{f(u)+f(v)}{2},\\
F^{cir}(u,v) &= \frac{f(u)+f(v)}{2} - \frac{|a((u+v)/2)|}{2}(u-v),\\
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- A basic requirement is the linear stability: if the conservative method is applied to the linear equation, it should be at least conditionally stable. For instance, the generalization of the centered method is not linearly stable and thus it is useless.

Finite volume methods

- It is easy to generalize these numerical fluxes to the general case:

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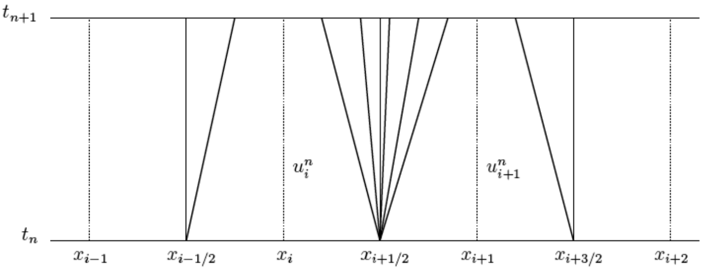
Godunov method

- Strategy of the method:



Godunov method

- Once the approximations at time $t = t_n$ have been computed, the Cauchy problem

$$\begin{cases} \dfrac{\partial \tilde{u}}{\partial t} + \dfrac{\partial}{\partial x} f(\tilde{u}) = 0, \quad x \in \mathbb{R},\ t > t_n, \\[2ex] \tilde{u}(x, t_n) = u^n(x), \quad x \in \mathbb{R}, \end{cases} \tag{23}$$

is considered, where u^n is again the piecewise constant taking value u_i^n at the cell I_i.

- The approximations at time t_{n+1} are computed then by averaging at every cell the solution of this Cauchy problem:

$$u_i^{n+1} = \frac{1}{\Delta x} \int_{I_i} \tilde{u}(x, t_{n+1})\, dx.$$

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- The solution of this Cauchy problem can be computed in terms of the solutions of a family of Riemann problems:

$$\begin{cases} \dfrac{\partial v}{\partial t} + \dfrac{\partial}{\partial x} f(v) = 0, \quad x \in \mathbb{R},\ t > 0, \\ \\ v(x,0) = \begin{cases} u_i^n & \text{if } x < 0; \\ u_{i+1}^n & \text{if } x > 0. \end{cases} \end{cases}$$

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can be easily computed.

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- If the CFL1/2 condition
$$\sup_{i,n} |a(u_i^n)| \frac{\Delta x}{\Delta t} \leq \frac{1}{2}.$$
is imposed, the waves issuing from $x_{i+1/2}$ at time t_n do not reach $x = x_i$ nor $x = x_{i+1}$ before time t_{n+1}.
- Therefore, $\tilde{u}$ is continuous at the points x_i and it is the solution of the Cauchy problem (23).
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- In effect, using that $\tilde{u}$ is a weak solution of the scalar law we have:

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- Exercise 5: Apply Godunov method to the linear conservation law

$$u_t + au_x = 0,$$

and check that it coincides with the upwind method.

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Approximate Riemann solvers

- If the computation of the solutions of the Riemann problems is difficult or very costly (this may be the case when f is not strictly concave or convex, and for systems), a possible strategy is to use an **approximate Riemann solver**: the idea is to use an approximation of the solution of the Riemann problems associated to every inter-cell.

$$\tilde{u}(x,t) = \widetilde{V}\left(\frac{x - x_{i+1/2}}{t - t_n}; u_i^n, u_{i+1}^n\right), \quad x \in [x_i, x_{i+1}],\ t \in (t_n, t_{n+1}].$$

- In the case of the linear Rieman solvers, $\widetilde{V}(x/t; u, v)$ is the self-similar solution of a linear Riemann problem:

$$\begin{cases} \dfrac{\partial \tilde{w}}{\partial t} + a(u,v)\dfrac{\partial \tilde{w}}{\partial x} = 0, \quad x \in \mathbb{R},\ t > 0, \\ \tilde{w}(x,0) = \begin{cases} v_L & \text{if } x < 0; \\ v_R & \text{if } x > 0; \end{cases} \end{cases}$$

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Linear approximate Riemann solvers

- Reasonning as in the case of the Godunov method, if the CFL condition

$$\sup_{i,n} \left| a^n_{i+1/2} \right| \frac{\Delta x}{\Delta t} \leq \frac{1}{2},$$

is satisfied, where

$$a^n_{i+1/2} = a(u^n_i, u^n_{i+1}),$$

it is possible to rewrite the method as follows:

$$\begin{aligned} u_i^{n+1} &= \frac{1}{\Delta x}\left(\int_{x_{i-1/2}}^{x_i} \tilde{v}(x, t_{n+1})\, dx + \int_{x_i}^{x_{i+1/2}} \tilde{v}(x, t_{n+1})\, dx\right) \\ &= \frac{1}{\Delta x}\left(\int_{x_{i-1/2}}^{x_i} \tilde{v}(x, t_n)\, dx + \int_{x_i}^{x_{i+1/2}} \tilde{v}(x, t_n)\, dx\right. \\ &+ \left.\int_{t_n}^{t_{n+1}} a^n_{i-1/2}\left(\tilde{V}(0; u^n_{i-1}, u^n_i) - u^n_i\right)\, dt - \int_{t_n}^{t_{n+1}} a^n_{i+1/2}\left(u^n_i - \tilde{V}(0; u^n_i, u^n_{i+1})\right)\, dt\right) \\ &= u^n_i + \frac{\Delta t}{\Delta x}\left(a^n_{i-1/2}\left(\tilde{V}(0; u^n_{i-1}, u^n_i) - u^n_i\right) + a^n_{i+1/2}\left(u^n_i - \tilde{V}(0; u^n_i, u^n_{i+1})\right)\right), \end{aligned}$$

Linear approximate Riemann solvers

- In the previous equality, it has been used that $\tilde{V}(x/t; u_i^n, u_{i+1}^n)$ is a solution of

$$\tilde{v}_t + a^n_{i+1/2}\tilde{v}_x = 0$$

in $[x_i, x_{i+1/2}] \times [t_n, t_{n+1}]$ and a solution of

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- Now observe that, if $a^n_{i+1/2} > 0$ then:

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- Therefore:

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- Analogously:

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Roe method

- Therefore, a numerical method base on a linear approximate Riemann solver can be written in the form:

$$u_i^{n+1} = u_i^n - \frac{\Delta t}{\Delta x}\left(a_{i-1/2}^{n,+}(u_i^n - u_{i-1}^n) + a_{i+1/2}^{n,-}(u_{i+1}^n - u_i^n)\right).$$

- This numerical method can be interpreted as a conservative method if the linearization $a(u,v)$ satisfies the Roe property:

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- Using the equalities:

$$a^+ = \frac{a + |a|}{2}, \quad a^- = \frac{a - |a|}{2},$$

we obtain:

$$\begin{aligned}
u_i^{n+1} &= u_i^n - \frac{\Delta t}{\Delta x}\left(a_{i-1/2}^{n,+}(u_i^n - u_{i-1}^n) + a_{i+1/2}^{n,-}(u_{i+1}^n - u_i^n)\right) \\
&= u_i^n - \frac{\Delta t}{2\Delta x}\left(a_{i-1/2}^n(u_i^n - u_{i-1}^n) + a_{i+1/2}^n(u_{i+1}^n - u_i^n)\right. \\
&\quad \left. + |a_{i-1/2}^n|(u_i^n - u_{i-1}^n) - |a_{i+1/2}^n|(u_{i+1}^n - u_i^n)\right) \\
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&= u_i^n + \frac{\Delta t}{\Delta x}\left(F^{Roe}(u_{i-1}^n, u_i^n) - F^{Roe}(u_i^n, u_{i+1}^n)\right)
\end{aligned}$$

Roe property

- where the numerical flux is given by:

$$F^{Roe}(u,v) = \frac{f(u)+f(v)}{2} - \frac{1}{2}|a(u,v)|(v-u).$$

- For scalar problems, the Roe property and the continuity determine the linearization $a(u,v)$:

$$a(u,v) = \begin{cases} \dfrac{f(u)-f(v)}{u-v} & \text{if } u \neq v; \\ f'(u) & \text{if } u = v. \end{cases}$$

- Roe method has very good shock-capturing properties: if the solution of the Riemann problem associated to the states u_L, u_R consists of a shock linking the states, then the solution of the approximate linear Riemann problem is the same.
- In particular, stationary shock waves are exactly captured.

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- Roe method has very good shock-capturing properties: if the solution of the Riemann problem associated to the states u_L, u_R consists of a shock linking the states, then the solution of the approximate linear Riemann problem is the same.
- In particular, stationary shock waves are exactly captured.

Roe method

- Nevertheless, Roe method has an important drawback: the numerical solutions can converge to a weak solution which is not the entropy solution of the system.
- This drawback can be overcome by using an entropy fix technique that prevents the numerical viscosity to vanish when one of the eigenvalues vanishes. For instance, the entropy fix technique proposed in Harten, Hyman 1983 consists in modifying the numerical flux as follows:

$$\widetilde{F}^{Roe}_{\varepsilon}(u,v) = \frac{f(u)+f(v)}{2} - \frac{1}{2}\left|a(u,v)\right|_{\varepsilon}(u-v),$$

where:

$$|a|_{\varepsilon} = |a| + 0.5\left\{\Big(1+\operatorname{sgn}(\varepsilon - |a|)\Big)\left(\frac{a^2+\varepsilon^2}{2\varepsilon} - |a|\right)\right\},$$

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- Exercise 7: Apply Roe method to the linear conservation law

$$u_t + au_x = 0,$$

and check that it coincides with the upwind method.

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Stability and convergence

- Lax-Wendroff's theorem does not ensure the convergence of conservative methods: it only characterizes the possible limits of the numerical solutions.
- In order to ensure the convergence, an adequate stability condition has to be required. caso de los problemas lineales.
- The right concept of stability for nonlinear problems is related to the total variation: let u^n be the piecewise constant function taking value u_i^n in I_i. The total variation of u^n is the quantity:

$$TV(u^n) = \sum_i |u_{i+1}^n - u_i^n|. \tag{24}$$

- A numerical method is said to be TV-stable if, for any initial condition u_0 and $T > 0$, there exist two positive constant $\Delta t_0, K$ such that:

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- A numerical method is said to be TVD (*Total Variation Diminishing*) if, for every u_0, Δt, Δx, one has:

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- Obviously, every TVD method is TV-stable.
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Numerical methods for systems of conservation laws

- Let us consider again a system of conservation laws:

$$\mathbf{u}_t + \mathbf{u}(\mathbf{u})_x = 0.$$

- A conservative numerical method can be written as follows:

$$\mathbf{u}_i^{n+1} = \mathbf{u}_i^n + \frac{\Delta t}{\Delta x}\{\mathbf{F}_{i-1/2}^n - \mathbf{F}_{i+1/2}^n\},$$

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  is the numerical flux.

- Most of the numerical fluxes introduced for scalar conservation laws can be easily extended to systems. This is the case of the Lax-Wendroff or the Lax-Friedrichs fluxes:

$$\begin{aligned}
\mathbf{F}^{LW}(\mathbf{u},\mathbf{v}) &= \frac{\mathbf{f}(\mathbf{u})+\mathbf{f}(\mathbf{v})}{2} - \frac{\Delta t}{2\Delta x}\mathcal{A}((\mathbf{u}+\mathbf{v})/2)^2(\mathbf{u}-\mathbf{v}),\\
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  represents the self-similar solution of the Riemann problem with initial data **u** and **v**. But it may be difficult to implement…

- Finally, the extension of Roe flux requires a **Roe linearization** of the Jacobian matrix $\mathcal{A}(\mathbf{u})$, i.e. a continuous matrix-valued function $\mathcal{A}(\mathbf{u},\mathbf{v})$ such that:
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- Here, $\mathcal{K}(\mathbf{u},\mathbf{v})$ is a matrix whose i-th column is an eigenvector associated to $\lambda_i(\mathbf{u},\mathbf{v})$ and $|\Lambda(\mathbf{u},\mathbf{v})|$ is the diagonal matrix whose coefficients are:

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- A Roe linearization is available for the homogeneous shallow water system. Given two states:

$$\mathbf{u}_i = \begin{pmatrix} h_i \\ q_i \end{pmatrix}, \quad i = 1, 2,$$

let us define:

$$\mathcal{A}(\mathbf{u}_1, \mathbf{u}_2) = \begin{bmatrix} 0 & 1 \\ -\bar{u}^2 + g\bar{h} & 2\bar{u} \end{bmatrix} \tag{25}$$

where

$$\bar{u} = \frac{\sqrt{h_1} u_1 + \sqrt{h_2} u_2}{\sqrt{h_1} + \sqrt{h_2}}, \quad \bar{h} = \frac{h_1 + h_2}{2}, \tag{26}$$

where

$$u_i = \frac{q_i}{h_i}, \quad i = 1, 2.$$

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$$|\mathcal{A}(\mathbf{u}_1, \mathbf{u}_2)|, \mathcal{A}^{\pm}(\mathbf{u}_1, \mathbf{u}_2).$$

Notice that the computation is trivial when both the eigenvalues have the same sign, i.e. for supercritical regimes (why?).

Numerical methods for systems of conservation laws

- A Roe linearization is available for the homogeneous shallow water system. Given two states:

$$\mathbf{u}_i = \begin{pmatrix} h_i \\ q_i \end{pmatrix}, \quad i = 1, 2,$$

let us define:

$$\mathcal{A}(\mathbf{u}_1, \mathbf{u}_2) = \begin{bmatrix} 0 & 1 \\ -\bar{u}^2 + g\bar{h} & 2\bar{u} \end{bmatrix} \tag{25}$$

where

$$\bar{u} = \frac{\sqrt{h_1}u_1 + \sqrt{h_2}u_2}{\sqrt{h_1} + \sqrt{h_2}}, \quad \bar{h} = \frac{h_1 + h_2}{2}, \tag{26}$$

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Numerical methods for balance laws

- In general, a numerical scheme is said to be well-balanced if it captures correctly the smooth stationary solutions of the system, or at least a family of them.
- In shallow water systems, the numerical schemes are usually required to preserve at least the solutions corresponding to water at rest: this is the so-called the C-property (Bermúdez & Vázquez, 1994).
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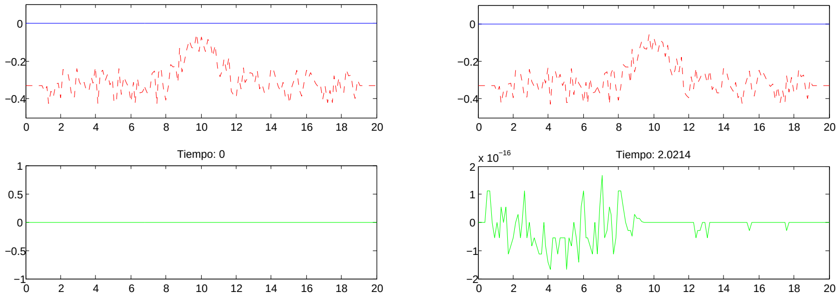
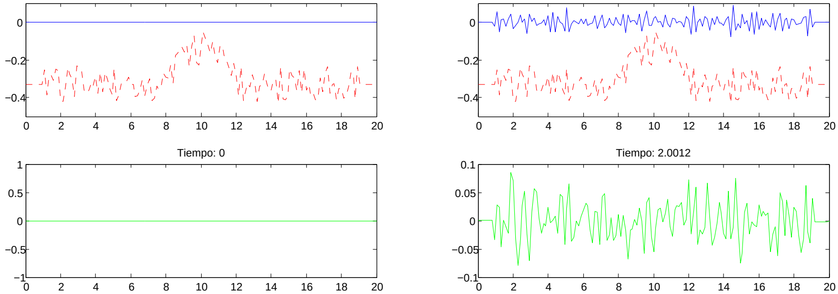
Well balanced method

Well-balanced method: bottom, water surface, discharge.



Well balanced method

A method whichi is not well-balanced method: bottom, water surface, discharge.



Water at rest solutions

- We consider the complete shallow water model:

$$\mathbf{u}_t + \mathbf{f}(\mathbf{u})_x = \mathbf{S}(\mathbf{u})\frac{dH}{dx},$$

$$\mathbf{u} = \begin{pmatrix} h \\ q \end{pmatrix}, \quad \mathbf{f}(\mathbf{u}) = \begin{pmatrix} q \\ q^2/h + gh^2/2 \end{pmatrix}.$$

$$\mathbf{S}(\mathbf{u}) = \begin{pmatrix} 0 \\ gh \end{pmatrix}.$$

- It can be easily shown that, given any constant C,

$$\mathbf{u}(x) = \begin{pmatrix} h(x) \\ q(x) \end{pmatrix} = \begin{pmatrix} C + H(x) \\ 0 \end{pmatrix}, \quad \forall x,$$

is a stationary solution of the system. This solution corresponds to water at rest with the free surface located at the plane $z = C$.

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- The values of this stationary solution at two different points x_1 and x_2 satisfy:

$$\mathbf{f}(\mathbf{u}(x_2)) - \mathbf{f}(\mathbf{u}(x_1)) = \mathbf{S}\left(\frac{1}{2}(\mathbf{u_1} + \mathbf{u_2})\right)(H(x_2) - H(x_1)). \tag{27}$$

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- We consider numerical methods of the form:

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Well-balanced property

- The numerical method is said to satisfy the C-property if it preserves water at rest solutions.
- More specifically, given any water at rest solution

$$h = H + C, \quad q = 0,$$

if we consider the initial condition

$$\mathbf{u}_i^0 = \begin{pmatrix} h(x_i) \\ q(x_i) \end{pmatrix} = \begin{pmatrix} H(x_i) + C \\ 0 \end{pmatrix}$$

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- Let us consider Roe numerical flux. An obvious choice for $S^{\pm}$ is:

$$S^{\pm}(\mathbf{u}, \mathbf{v}) = \frac{1}{2}\mathbf{S}\left(\frac{1}{2}(\mathbf{u} + \mathbf{v})\right),$$

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Upwind discretization of the source term

- We propose the definitions:

$$S^{\pm}(\mathbf{u},\mathbf{v}) = \mathcal{P}^{\pm}(\mathbf{u},\mathbf{v})\mathbf{S}\left(\frac{1}{2}(\mathbf{u}+\mathbf{v})\right).$$

- The corresponding numerical method satisfies the C-property. In effect, the numerical method can be equivalently written as follows:

$$\begin{aligned}\mathbf{u}_i^{n+1} &= \mathbf{u}_i^n - \frac{\Delta t}{\Delta x}\Big(\mathcal{A}_{i-1/2}^{n,+}\cdot(\mathbf{u}_i^n - \mathbf{u}_{i-1}^n) + \mathcal{A}_{i+1/2}^{n,-}(\mathbf{u}_{i+1}^n - \mathbf{u}_i^n)\\ &+ S_{i-1/2}^{n,+}(H_i - H_{i-1}) + S_{i+1/2}^{n,-}(H_{i+1} - H_i)\Big).\end{aligned}$$

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Upwind discretization of the source term

- Finally, using the Roe property and the equality (27), we obtain:

$$
\begin{aligned}
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&= \mathbf{u}_i^0,
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as we wanted to prove.

Upwind discretization of the source term

- It is possible to derive well-balanced numerical methods that use any numerical flux: Audusse, Bouchut, Bristeau, Klein, Perthame 2004, Castro, Pardo, CP , Toro, 2011...
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Extension to high order

- Let us consider a reconstruction operator that, given a sequence of cell values $\{\mathbf{u}_i\}$, provide a smooth function at every cell

$$P_i(x; \mathbf{u}_{i-l}, \ldots, \mathbf{u}_{i+r})$$

that depend on the values at the neighbor cells $\{I_k\}_{k=i-l}^{i+r}$ that constitute the *stencil*, in such a way that:

  - It is conservative, i.e.

$$\mathbf{u}_i = \frac{1}{\Delta x} \int_{x-i-1/2}^{x_{i+1/2}} P_i(x; \mathbf{u}_{i-l}, \ldots, \mathbf{u}_{i+r})\, dx.$$

  - If the cell values are the averages of a smooth function $\mathbf{u}(x)$ then

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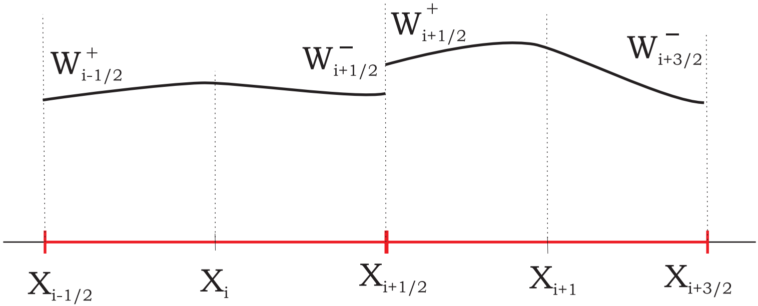
  - It is conservative, i.e.

$$\mathbf{u}_i = \frac{1}{\Delta x} \int_{x-i-1/2}^{x_{i+1/2}} P_i(x; \mathbf{u}_{i-l}, \ldots, \mathbf{u}_{i+r})\, dx.$$

  - If the cell values are the averages of a smooth function $\mathbf{u}(x)$ then

$$\mathbf{u}(x) = P_i(x; \mathbf{u}_{i-l}, \ldots, \mathbf{u}_{i+r}) + O(\Delta x^p), \quad \forall x \in [x_{i-1/2}, x_{i+1/2}].$$

Reconstruction operator



Reconstruction operator

- Let us suppose that the depth function H is continuous. We consider the semidiscret numerical method:

$$\frac{d\mathbf{u}_i}{dt} = \frac{1}{\Delta x}\left(\mathbf{F}_{i-1/2} - \mathbf{F}_{i+1/2} + \int_{x_{i-1/2}}^{x_{i+1/2}} S(P_i^t)H'(x)\,dx\right),$$

- where

$$P_i^t(x) = P_i(x; \mathbf{u}_{i-l}(t), \dots, \mathbf{u}_{i+r}(t)),$$

$$\mathbf{u}_{i+1/2}^{+} = P_{i+1}^t(x_{i+1/2}), \quad \mathbf{u}_{i+1/2}^{-} = P_i^t(x_{i+1/2}),$$

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- The numerical method is well-balanced for water at rest if the reconstruction operator is exact for constant functions and the reconstruction is computed as follows:
  - First, the reconstruction operator is applied to the variables q and $\eta = h - H$ to obtain $p^t_{q,i}$, $p^t_{\eta,i}$.
  - Then, the reconstruction of the variable h is defined by $p^t_{h,i} = H + p^t_{\eta,i}$.
- The semidiscrete scheme is fully discretized by using an adequate time-stepping, for instance TVD-RD methods.
- The integral terms can be computed with a quadrature formula with an order of accuracy greater or equal than p.
- Examples of reconstruction operators: MUSCL(Van Leer 1979), PPM Colella, Woodward 1984), ENO (Harten, Engquist, Osher & Chakravarthy, 1987), WENO (Liu, Osher & Chan, 1994, Jiang & Shu, 1996), PHM (Marquina, 1994), etc.
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Extension to 2d problems

- The 2d problem can be written in the form:

$$\mathbf{u}_t + \mathbf{f}_1(\mathbf{u})_x + \mathbf{f}_2(\mathbf{u})_y = S_1(\mathbf{u})H_x + S_1(\mathbf{u})H_y,$$

- where

$$\mathbf{u} = \begin{pmatrix} h \\ q_1 \\ q_2 \end{pmatrix}, \quad \mathbf{f}_1(\mathbf{u}) = \begin{pmatrix} q_1 \\ \dfrac{q_1^2}{h} + \dfrac{g}{2}h^2 \\ \dfrac{q_1 q_2}{h} \end{pmatrix}, \quad \mathbf{f}_2(\mathbf{u}) = \begin{pmatrix} q_2 \\ \dfrac{q_1 q_2}{h} \\ \dfrac{q_2^2}{h} + \dfrac{g}{2}h^2 \end{pmatrix},$$

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Finite volumes

- The domain D is decomposed into subsets (closed polygons) called cells of finite volumes, $V_i \subset \mathbb{R}^2$;
- $N_i \in \mathbb{R}^2$ is the centre of V_i;
- $\mathcal{N}_i$ is the set of indexes j such that V_j is a neighbor of V_i;
- E_{ij} is the common edge to two neighbor cells V_i and V_j, and $|E_{ij}|$ represents its length;
- $\eta_{ij} = (\eta_{ij,1}, \eta_{ij,2})$ is the normal unit vector of the edge E_{ij} pointing towards the cell V_j;

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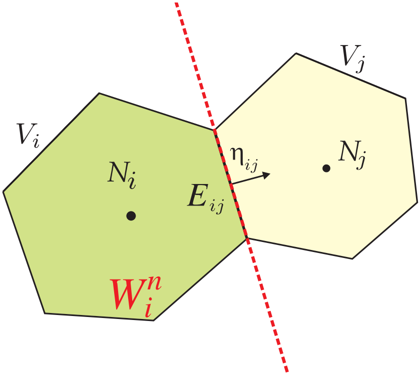

Finite volumes

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$$\mathbf{u}_i^{n+1} = \mathbf{u}_i^n - \frac{\Delta t}{|V_i|}\left(\sum_{j\in\mathcal{N}_i} |E_{ij}|\left(F_{ij} - S_{ij}^-\left(H_j - H_i\right)\right)\right)$$

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$$F_{ij} = \eta_{ij,1}F_1(\mathbf{u}_i^n, \mathbf{u}_j^n) + \eta_{ij,2}F_2(\mathbf{u}_i^n, \mathbf{u}_j^n);$$

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